On Pendulums and Air Resistance

The mathematics and physics of Denis Diderot

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Abstract. In this article Denis Diderot's Fifth Memoir of 1748 on the problem of a pendulum damped by air resistance is discussed in its historical as well as mathematical aspects. Diderot wrote the Memoir in order to clarify an assumption Newton made without further justification in the first pages of the *Principia* in connection with an experiment to verify the Third Law of Motion using colliding pendulums. To explain the differences between experimental and theoretical values, Newton assumed the bob was retarded by air resistance expressible in terms of a force F_R proportional to arc traversed. By giving Newton's arguments a mathematical scaffolding and recasting his geometrical reasoning in the language of differential calculus, Diderot provided a step-by-step solution guide to the problem. He also showed that Newton's assumption was equivalent to having assumed F_R proportional the bob's velocity v, when in fact he believed it should be replaced by $F_R \sim v^2$. His solution is presented in full detail and his results are compared to those obtained from a Lindstedt-Poincaré approximation for an oscillator with quadratic damping. It is shown that, up to a prefactor, both results coincide. Some results that follow from his approach are presented and discussed for the first time. Experimental evidence to support Diderot's or Newton's claims is discussed together with the limitations of their solutions. Some misprints in the original memoir are pointed out.

1 Introduction

The last things to come to mind when one thinks of Denis Diderot (1713–1784) are the field of mathematics and physics. Rightly regarded as one of the most prolific minds of the 18th century, his name evokes first and foremost the emblematic *Encyclopédie des Sciences, des Arts e des Métiers*, of which he was the main editor and to which he dedicated 25 years of his life[1]. Given the gargantuan range of his interests – encyclopedic in the broadest sense of the word – one can find under his pen works of philosophical inquiry, historiography, critique of art, novels and translations. In his

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[1] Diderot's association with the *Encyclopédie*'s started in 1747. The first 17 volumes of the colossal work were published between 1751 and 1765. The eleven extra volumes of plates were finished by 1772.

seminal book on the Enlightenment, Peter Gay (Gay 1966) affirms that *'... Diderot was, with almost equal competence, translator, editor, playwright, psychologist, art critic and theorist, novelist, classical scholar, and educational and ethical reformer'.* Thus it should come as no surprise that the specialized and non-specialized literature on Diderot reflect, in variety and extent, the breadth of his intellectual production. However, given the wealth of mathematical and physical problems Diderot tried his hand at, the same cannot be said of his mathematical treatises. With a few praise-worthy exceptions (Ballstadt 2008; Coolidge 1949; Krakeur and Krueger 1941) this facet of Diderot – that of the mathematician – remains largely unexplored. However, as Ballstadt convincingly argues, Diderot's relation to science can only be understood if one understands the role mathematics played in his thought.

One possible reason for this lack of knowledge about mathematics in Diderot's life could be the fact that his association with other fields of inquiry is so vast that his mathematical exploits are to a great degree still regarded as a minor diversion from other more relevant contributions. Even if we cannot compare him with contemporary mathematicians like Jean le Rond d'Alembert (1717–1783), his co-editor at the Encyclopedia until 1759, his involvement with mathematics should not be regarded as less serious. From 1733 to 1761 Diderot worked on problems related to sundials, probability theory, calculus of annuities, deciphering machines and algebraic curves. He had plans to publish commentaries on Newton's *Principia* (but was forestalled by Jacquier and Le Seur's annotated edition), and worked on problems of applied mathematics, from pendulums to vibrating strings. His later and less technical writings are interspersed with comments on mathematics and the value he accorded to this field of knowledge can be judged by the fact that he placed it at the basis of a curriculum that he prepared for a university in Russia[2]. His views on mathematics can be epitomized in the paragraph he wrote for his *Premières notions sur les mathématiques*, an introductory mathematics textbook which went unpublished and were discovered quite recently (Diderot 1975):

> *Mathematics encompasses almost all fields of human knowledge. It helps one differentiate right from wrong, to convince the spirit of truths already known, to unveil new ones and lead with certitude all sciences man acquires on reason alone to perfection.*

He was what we could call today an amateur, but to that one should add that he was an extremely competent one. He did original research, was technically proficient and conversant with contemporary mathematical research.

The first to consider mathematics in the context of Diderot's works were L.G. Krakeur and R. L. Krueger (Krakeur and Krueger 1931). A more technical analysis was undertaken by J. Coolidge (Coolidge 1949) and despite its shortcomings – only some of Diderot's works are analysed and then only partially – it remains a starting point and a valuable source of information for any serious study. Ballstadt considered the role mathematics played in Diderot's views on natural philosophy and, given his encyclopedic bent of mind, which encompassed basically every branch of knowledge at that time, mathematics can be rightfully said to have been one of his greatest and never fading passions. Diderot himself referred in his later years to his relation to mathematics in a rather amusing way. In his *Réfutation d'Helvétius* (Diderot 1774) he tells the story of his parents who, upon noticing that their first-born son was rather

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[2] His *Plan d'une université pour le gouvernement de Russie* (1775) was a personal request of the Russian Empress Catherine II (Wilson 1972, p. 18), at whose court Diderot spent the Winter of 1773–74.

predisposed to studies, decided to send him to the local provincial school and later to Paris where (Diderot 1774, p. 580)[3]:

> *They gave him texts on arithmetic, algebra and geometry, which he devoured. Later, admonished to [devote himself] to more agreeable studies, he found pleasure in the reading of Homer, Virgil, Tasso and Milton, but always returned to mathematics, just like an unfaithful husband who, tired of his mistress, returns from time to time to his wife.* [4]

For Ballstadt mathematics was the only branch of 18th century science in which Diderot can be said to have been a practitioner (Ballstadt 2008). Given his standing for the *Siècle des Lumières* and his intellectual acumen, this assessment carries considerable weight. Thus, it is the purpose of this article to expand some of the previous works by analyzing Diderot's works from a more technical perspective while simultaneously highlighting the didactic approach Diderot chose to present them. To the best knowledge of the author, this facet of Diderot's works has not been explored.

In the present work Diderot's Fifth Memoir on the harmonic oscillator with quadratic damping is discussed. It is a problem that still draws the attention of physicists and mathematicians (Linz 1995; Cvetićanin 2009) and a better understanding of these two timely separated vistas comes at a cost: the long calculations of Diderot's treatise and the modern view on the subject cannot be presented in a condensed way. With this in mind, this article was written in a way to allow the reader to examine those approaches independently, if necessary, without compromising the understanding of Diderot's work. The paper was thus organized as follows: in Section 2, a brief description of the known mathematical works of Diderot is given. This is followed by Section 3, where the historical context of Diderot's Memoir on the damped pendulum is discussed. The memoir itself is treated in Section 4. It starts by introducing Newton's discussion of the colliding pendulums. Diderot's own solution is presented next in all its mathematical detail. In order to make Diderot's calculations more transparent to the modern reader, his notation is explained and some misprints in the original are pointed out. The modern approach is left for Section 5 where the pendulum with arbitrary swing amplitudes is discussed. The effect of air resistance on the pendulum's movement is also discussed and an approximate solution using a Lindstedt-Poincaré expansion is presented. Experimental support for a quadratic drag, *i.e.* $F_R \sim v^2$ is presented. This section can be read independently and may be skipped by those interested only in the Diderot's solution. Some of the results presented in this section are important for a better evaluation of Diderot's Memoir, but these are clearly pointed out when necessary. In Section 6 some conclusions are drawn. The paper closes with an Appendix, where the various quotes by Diderot and others are reproduced in their original form.

2 Diderot and Mathematics

In 1748, Pissot and Durant of Paris published an octavo volume with the unassuming title *Mémoires sur différens sujets de Mathématiques.* With its deluxe format and exquisite engravings, it was *"... one of the most coquettish [volumes] that was ever published on such arid subjects"*, as Maurice Tourneaux remarked (Wilson 1972, p.

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[3] All texts by Diderot and some of his sources are given in the Appendix in their original language.

[4] Diderot changed the story a bit: he actually tried to leave Langres, his hometown, secretly, and join the Jesuits in Paris. His father got wind of it and took his son himself to Paris, enrolling him at the *Collège d'Harcourt* (Wilson 1972, p. 22).

89) [5]. The book contained five different treatises on different subjects, mostly applied mathematics, and are the only ones published during his lifetime. For the sake of completeness, they are described in what follows.

1. The first and longest memoir is a study entitled *Principes généraux d'acoustique*. It deals with the general properties of sound waves and their relation to vibrating strings, the mathematics of pitch and the use of logarithms in the production of harmonious sounds. Coolidge regards it as the most important of all five memoirs (Coolidge 1949) and, from a historical point of view, this assessment is quite accurate: Diderot was, inadvertently or not, flerting with what would become one of the greatest controversies of 18th century mathematics: the so-called *vibrating string controversy* (Truesdell 1961, Ravetz 1961, Kleiner 1989, Wheeler and Crummett 1984). The main actors were d'Alembert, Leonhard Euler (1707–1783), Daniel Bernoulli (1700–1782), Joseph-Louis Lagrange (1736–1813) and Joseph Fourier (1768–1830).

Musical theory was taking big strides with Euler's *Tentamen novae theoriae musicae* of 1739 and Jean-Philippe Rameau's *Traité de l'harmonie*, published in 1722. Rameau (1683 – 1764) was a contemporary of Diderot, whose interest in music was common for his time. From a mathematical point of view the main problem was to understand the relation between the vibration of strings and the sounds they produced. It was clearly formulated: an elastic string with ends fixed at $x = 0$ and $x = l$ is deformed into some initial shape and released to vibrate at time $t = 0$. The problem is to determine the function that describes its shape for arbitrary t. The controversy was centered around what one meant by 'function' and the struggle between two mental images: on the one hand the geometric image in the form of curves and on the other the algebraic image, a formula with infinitely many terms. There reason why the concept took so long to develop was in part due to the lack of prerequisites (Kleiner 1989): the extension of the concept of number on the real line and the lack of symbolic algebra. It was precisely this problem that led d'Alembert to the discovery of the one dimensional wave equation in 1746 (Cannon and Dostrovsky 1981). As with all mathematicians before him, Diderot was greatly influenced by Brook Taylor's article *On the motion of a taut string* (Taylor 1713). Diderot's article draws strongly from Taylor's work, a piece of physics written in a rather cryptic way. For the modern reader the way natural scientists attacked the problem can be quite puzzling if one does not consider the two main propositions which were at the base of any discussion of the problem. Diderot's devotes a great part of his treatise to a detailed discussion of these propositions, namely

(i) *isochronism* of movement, that is the frequency of a periodic motion is independent of the amplitude of motion. This can be summarized in what came to be known as Mersenne's law, which relates the frequency ν of a vibrating string to its length l, the tension T and its linear mass density μ:

$$\nu \ \propto \ \frac{1}{l}\sqrt{\frac{T}{\mu}} \tag{1}$$

Much of the discussion on the subject was aimed at finding the exact proportionality constant in the expression above.

(ii) Another important assumption in the case of the string was that of the *simultaneous crossing of the axis*. It was observed that the simplest motions occurred when all parts of a taut string crossed their equilibrium position simultaneously. This

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[5] Maurice Torneaux and Jules Assézat were the first publishers of Diderot's complete works.

is of course correct if one considers no modes of vibration other than the normal ones. It becomes however a major problem when one tries to explain the existence of overtones.

D'Alembert published his first study on the subject in 1747, a work which Diderot does not fail to mention (d'Alembert 1747). Unfortunately for him however was the fact that right after his memoirs were printed the discussion acquired momentum and a heated debate started, a debate which would only be settled 60 years later with Fourier's work (Fourier 1807; Cannon and Ostrovsky 1981).

2. The memoir *Examen de la développante du cercle* is a treatise on involutes. An involute is the curve obtained by the free end of a taut string attached to a curved body as it unwinds from that body [6]. In spite of reading like a piece of pure mathematics, Diderot's motivation seems to have been a practical one: whether it would be possible to draw curves without recurring to a ruler and a compass or not. In other words, he was looking for some kind of device with which one could draw 'mechanical curves' (*courbes mécaniques*) [7]. More importantly, in this memoir takes up on his hobbyhorse, the squaring of the circle, since involutes are intimately connected with the rectification problem: given some region of space delimited by known curves, one expects to find its area by transforming these curves into straight ones with the help of involutes. Involutes were introduced by Christiaan Huygens (1629 – 1695) in his treatise on pendulums and clockmaking, the *Horologicum Oscillatorium* (Huygens 1673) They are also relevant in the design of mechanical gears, since dents which involute profiles have a better distribution of forces and are less prone to noise and wear, as first noted by Leonhard Euler (1707 – 1783) (Krakeur and Krueger 1941). There is no direct mention of Huygens in Diderot's writings. However, involutes in association with synchronous pendulums are mentioned in Fathers Le Seur and Jacquier's comments on the *Principia* (Le Seur and Jacquier 1739), a work Diderot had studied. memoir. In his voluminous work on plane curves, G. Loria mentions Diderot's work as the one where a methodical treatment of the involute of a circle is given (Loria 1930).

3. The short 5-page Memoir *Examen d'un principe de mécanique sur tensions des cordes* presents an experiment proposed by Diderot to decide a question posed by the Italian mathematician Giovanni Borelli (1608 – 1679): imagine a rope whose one end is attached to a fixed point, while a weight A hangs at the other. Will the replacement of the fixed point by an equal weight change the resulting tension? The answer which today one may find in any elementary physics book was, at the time, an open question (Krakeur and Krueger 1941).

4. The memoir *Projet d'un novel orgue* is related to the first one. It is a project for an organ that could be played even by those who have no musical training and was based on the use of a punch card of sorts. This work had been published separately the year before in the *Mercure de France* (Krakeur and Krueger 1941) and received mixed reviews: the Jesuit *Journal de Trévoux*, while praising the author, ridiculed the project as unpractical and unnecessary (Mayer 1975). This prompted Diderot to defend it in

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[6] An example is the spiral described by the tip of a rope as one unwraps it from around a circle. In mathematics involutes are also called 'evolvents'.

[7] Today one speaks of 'algebraic' ('geometric' in Diderot's language) and 'transcendental' curves. Algebraic curves can be defined as the set of points given by the equation $f(x, y) = 0$ where $f(x, y)$ is polynomial in x and y. Transcendental curves intercept some straight line in an infinity of points and cannot be represented by a polynomial equation of finite degree.

the entry *Androïde* of the Encyclopédie. Jean-Jacques Rousseau (XXXX), who was responsible for musical matters in the Encyclopédie, mentioned it critically in the entry *Chronomètre*. Actually several of the topics treated in the Memoirs were used later by Diderot and D'Alembert in the Encyclopédie. It is rather amusing to read in this work some of the things Diderot himself mentioned as possible 'inconveniences' of his project: the ignorance of the author about the subject matter and the fact that organists could not be mediocre. As to his own ignorance, it should be remembered that Diderot knew personally professionals of the musical *millieu* as Rameau and he always set the bar rather high, whatever the subject he was dealing with.

5. The last memoir is the *Lettre sur la résistance de l'air au mouvement des pendules.* This memoir is the subject of the present article and as such will be discussed in more detail in Section 3.

In 1761 Diderot wrote three more essays under the title *Noveaux Mémoires sur Différents Sujets de Mathématiques*, but which were kept private and published only posthumously (Diderot 1761):

1. An article on the cohesion of bodies. Diderot tried to argue, without presenting any calculations, that an inverse-square law of attraction could be responsible for the cohesion of macroscopic matter. This article is more a defense of some propositions Newton brought forth in his *Principia*.

2. An article on the use of probability in calculating betting odds in the famous Saint Petersburg problem. In this article he corrected an error committed by d'Alembert and to be found in his discussion of *Croix ou Pile* in the Encyclopédie (d'Alembert 1784). The whole idea is the following: suppose a casino offers a player the chance of winning on the toss of a fair coin. The rule is the following: the casino puts 2 dollars on a pot. If the player, on his first trial, tosses a tail, he takes the 2 dollars and walks away. If he tosses a head, the casino doubles the amount in the pot to 4 dollars. Every time the player gets a head, the amount in the pot is doubled. After k consecutive tosses, if all of them were heads, the amount in the pot is $2^{(k+1)}$ dollars. In short the player wins $2^{(k+1)}$ dollars if he throws k consecutive heads. The moment he gets a tail, the casino pays him what is in the pot and the game is over. The question is: how much should the casino charge the player to enter the game? The problem is that, if one considers the expectation value one player is supposed to get, the sum is divergent, since the player has a probability $1/2$ of winning 2 dollars, $1/4$ of winning 4 dollars and so on. In other words, the expected value E is

$$E = \frac{1}{2} \times 2 + \frac{1}{4} \times 4 + \frac{1}{8} \times 8 + \cdots = 1 + 1 + 1 + \cdots = \infty \qquad (2)$$

Assuming the casino has unlimited resources, this result says that, considering what one expects to get, one should pay any price required to enter the game. The paradox consists exactly in the discrepancy between what one expects to get and what one is willing to pay. D'Alembert argued that no one in his right mind would pay any amount to enter such a game because there is a difference between what the calculations showed (which he thought wrong) and what common sense dictated. For d'Alembert once you get a sequences of heads, the probability of getting another one when you toss again is not $1/2$, that is, independent of the previous tosses, but should be smaller. Diderot argued against it and corrected his argument. The literature on this problem is rich (Daston 1979; Swijtink 1986) and, to my best knowledge, the most thorough analysis – historical and mathematical – of this problem is to be found in Samuelson (Samuelson 1977). The key point in the discussion is the concept of

utility, as understood in Economics: to paraphrase Samuelson, "the dollar I win is not as worth as the dollar I lose."

3. The third unpublished article is also a question on probability in general and smallpox inoculation in particular. It was a response to d'Alembert's criticism, who thought very sceptically about the use of probabilities in matters of life and death. For d'Alembert the mere possibility of getting infected and developing the disease via inoculation was reason enough not to get the vaccine at all. Diderot's involvement with 'political arithmetic' or probability theory was much in line with his engagement on public issues and was more philosophical than mathematical in nature.

Further works involve questions as varied as were Diderot's interests. There is a long article on cyclometry (squaring of the circle) and a project of a deciphering machine. Unfortunately the later survived only in fragmentary form and it is impossible to make out what Diderot's machine looked like and what it could accomplish, if ever built. There is a discussion on the 'geometry of infinity', which is basically a short description of the infinitesimal limit on a curve. In probability theory he just did not confine himself to games of chance or inoculations. He also calculated annuities for insurance purposes and outlined a money lottery for the Military School. He also wrote a very short comment on celestial mechanics, the duration of human life. From all these posthumous works, the most extensive is a manual of basic arithmetic for children that could, as Diderot said, be also used by *jeune mademoiselles*. This book was probably a product of Diderot's years as a teacher of mathematics to the children of wealthy families. The most up-to-date edition of his extant mathematical works can be found in the extensively annotated *Oeuvres Complètes* (Diderot 1975).

3 The Fifth Memoir on Newton and Colliding Pendulums

Judged by its title, the fifth and last memoir of 1748 seems to be one of those simple paradigmatic exercises to be found in most physics textbooks: to determine the retardation on the movement of a pendulum caused by air resistance. This apparent simplicity is deceiving and the problem of drag still draws the attention of contemporary researchers (Linz 1995; Cvetićanin 2009). On the one hand, the physics of retarded pendulums was crucial in the context of clock design (Huygens 1673; Schulze 1782; Wagner 1867; Matthews, Gauld and Stinner 2005). Pendulums still are a commonly employed device for measuring Earth's gravitational field g [8]. On the other hand, its history is closely connected with the history of Hydrodynamics, whose main actors were Euler, d'Alembert and Bernoulli. Interestingly enough, these were the same main actors of the problems Diderot treated in his first (acoustics) and second (curves) memoirs. The pendulum's importance is attested by the numerous *Mémoires* published in the years after Diderot's death, from the 1835 treatise of Giovanni Plana (1781–1864) to the 2-Volume *Mémoire* of Charles Wolf (Plana 1835; Wolf 1889; Smith 1998; Darrigol 2005). Despite its importance, a more careful look into the contents of the Fifth Memoir reveals, however, that Diderot does not seem to have been interested in Hydrodynamics itself, but in giving a didactic explanation of a commentary made by Newton in the first pages of the *Principia*, which Diderot goes as far as transcribing from in the original Latin [9].

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[8] Even if not precise as superconducting gravimeters, which can detect variations of order $10^{-11}\ ms^{-2}$ in the gravitational field (Virtanen 2006), pendulums are easy-to-get, affordable gravimeters.

[9] Diderot was known to be an accomplished latinist. He says he learned English by translating works in these language into French with the help of a English-Latin dictionary (Wilson 1972, p. 49).

The passage he quotes is concerned with is the experimental verification of *Actio = Reactio*, or the Third Law of Motion. This problem, which at a first might look rather far removed from the problem of damped oscillators, is actually the key to Diderot's assessment of Newton: to verify the validity of the Third Law, Newton conducted a series of collision experiment with two pendulums. But, as Newton observes, *… to bring this experiment to an accurate agreement with the theory, we are to have due regard as well to the resistance of the air as to the elastic force of the concurring bodies* (Newton 1687, p. 20). Newton however did not bother to say how exactly the air resistance should be, except that the retardation was proportional to the arc of the trajectory. Diderot showed that this assumption was equivalent to Newton having implicitly assumed a v-type law when in fact – so he thought – he should have favored a v^2-law. The way Diderot treated the problem and organized his article around Newton's ideas shows how competent he was in his assessment of the great master. Diderot knew the didactic value of mathematics and filled in the gaps Newton left open, or at least not explicitly written.

Diderot's motivation might have been quite personal: if one takes his dedicatory introduction at face value[10], he was asked to clarify a passage in the *Principia* (Diderot 1748, p. 320):

> *If the place [in the Principia] where Newton calculates the resistance caused by air on the movement of the pendulum embarrasses you, do not let your self-esteem be afflicted by it. As the greatest geometers will tell you, one encounters, in the depth and laconism of the Principia, [enough] reasons to completely console a man of penetrating mind who had some difficulty in understand them; and you will see shortly that there is another reason that seems even better to me – that the hypothesis this author started with might not be exact.*

The error Diderot is talking about is Newton's choice of a force linear in v. Previous works on the subject have given emphasis to this v vs. v^2 controversy (Krakeur and Krueger 1941, Coolidge 1949), when the truth is that Newton actually considered both types of force in the *Principia*. The first 31 propositions of Book II are dedicated to the problem of damped pendulums (Gauld 2009, 2010; Darrigol and Frisch 2009; Smith 1998) Actually Newton went so far as to use the more general expression $F_R(v) = a\ v + b\ v^{\frac{3}{2}} + c\ v^2$ in order to fit the results of experiments he conducted himself. At the end of Book II, Section I he concludes (Newton 1687, p. 165):

> *However, that the resistance of bodies is in the ratio of the velocity, is more a mathematical hypothesis than a physical one. In mediums void of all tenacity, the resistance made to bodies are as the square of the velocities. For by the action of a swifter body, a greater motion in proportion to a greater velocity is communicated to the same quantity of the medium in a less time; an in an equal time, by reason of a greater quantity of the disturbed medium, a motion is communicated as the square of the ratio greater; and the resistance (by Laws II and III) is as the motion communicated.*

Not surprisingly friction forces F_R proportional to v^2 are now called in the specialized literatura *Newton Friction*, whereas a v-dependent F_R is called *Stokes Friction*.

Newton's approach to the damping problem was criticized by Leonhard Euler and Daniel Bernoulli (1700–1782) for its lack of rigour, something which certainly did

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[10] The introduction is dedicated to M^{***}, whose identity is unknown. The whole volume of Memoirs is dedicated most probably to Marie Anne Victoire Pigeon d'Osangis (1724 – 1767), a french mathematician known by the name of Madame de Prémontval, as she was the wife of Pierre Le Guay de Prémontval (1716 – 1764), also a mathematician. M^{***} could have been just a fictive addressee as this Memoir in written in the form of a letter.

not baffle a practical mind like Diderot's as much as it baffled those of the great hydrodynamicists. Diderot's respect for Newton was too great: *'I have for Newton all deference one accords to the unique men of his kind'* (Diderot 1748, p. 332). So Diderot might have been motivated by something other than someone's request: he seems to have had the intention of publishing his own commentaries on the *Principia*, but he was superseded by the famous annotated edition of the Franciscan Fathers François Jacquier (1711–1788) and Thomas Le Seur (1703–1770) which came out between 1739 and 1742. So we read from his introduction (Diderot 1748, p. 320-321):

> *Something surprises me however: that you were advised to seek me in order to free you from your embarrassment. It is true that I studied Newton with the purpose of elucidating him. I should even tell you that this work was pushed on, if not successfully, at least with great vivacity. But I did not think of it any longer since the Reverend Fathers Le Seur and Jacquier made their commentaries public, and I did not feel tempted to ever reconsider it. There was, in my work, a few things you would not find in the work of these great geometers and a great many things in theirs you most surely would not find in mine. What do you ask of me? Even though mathematical matters were once much familiar to me, to ask me now about Newton is to talk of a past dream. However, to persevere in the habit of pleasing you I will leaf through my abandoned drafts, I will consult the sagacity of my friends and tell you what I can learn from them, telling you also, with Horace: if you can make these better, please let me know. If not, follow them with me.*

The Fifth Memoir might have also been part of a more general work: given that Diderot also wrote an article on involutes (Second Memoir) and these are intrinsically connected with the problem of constructing an isochronous pendulum, these two memoirs might bear some relation with Christiaan Huygens epochal *Horologium Oscillatorium*. It would well fit the interests of Diderot in the 'applied arts' and the fact that earlier in his career he prepared the general formulas and mathematical tables for a treatise of Antoine Deparcieux (1703 – 1768) on sundials [11]. Timekeeping devices could have exerted a certain fascination on him (Wilson 1972, p. 69). Tempting as this supposition might be, there is to the author's knowledge no mention of Huygens in Diderot's works. This does not mean that he did not know it, as Diderot was very well acquainted with the mathematical literature of his times. The memoirs of 1748 themselves might also have been an attempt of the author to appear more serious to the eyes of his contemporaries: by that year he had acquired a rather scandalous reputation with the publication of a rather brazen novel entitled *Les Bijoux Indiscrets*. It was published anonymously, but the author's name was no secret. The strongest evidence of Diderot's attempt to look serious can be read off from the opening phrase of his Memoirs, drawn from Horace's Satires (Diderot 1748, p. 232): '*[Sed tamen] amoto quaeramus seria ludo*', which roughly translates as 'plays aside, let us turn to serious matters.'

## 4 The Mathematical Pendulum from Diderot's Perspective

To understand Diderot's approach, one has to first consider Newton's experiment described in the initial pages of the *Principia*. This is the more so if one realizes that Diderot wrote his article as some sort of solution's manual to the arguments Newton expounded. The path Diderot chose to arrive at answers to the questions he poses at

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[11] It is thus not surprising that the entry *Cadran Solaire* (Sundial) in the *Encyclopédie* was signed by him and D'Alembert.

the beginning of his memoir seems at first rather awkward. But once one realizes that he is following Newton's logic closely, translating his arguments into mathematical form, one understands why he chose to solve the problem the way he did.

4.1 Newton's Solution

Newton's discussion is based on the experimental setup depicted in Fig. 1 below. It is reproduced in the section *Axioms, or Laws of Motion* of the *Principia*. It was inspired on earlier experiments done by Edme Mariotte (1620–1684) and Christopher Wren (1632–1723) on the collision of pendulums.
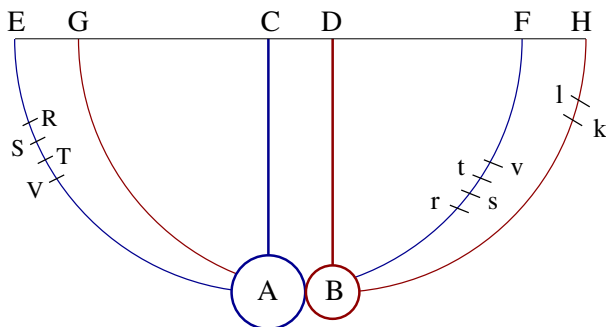
Fig. 1. The pendulum Newton considered in his experiment to prove that Action equals Reaction: As bob A is let loose from a given point R, it hits B. As a consequence of their collision, both move upwards, A reaching s and B reaching k. The different points of the trajectories marked r, t, v etc. and what they represented are discussed in the text. Diderot reproduced this picture in his memoir on the damped pendulum.

Newton wants to study the transfer of momentum between colliding pendulums. The collision happens at point A. Given that the bob will have a lower velocity at A as compared to what it would in vacuum, one has to correct for the lost momentum. To find this, Newton devises a simple trick: since this difference is proportional to the path traversed (see Fig. 1), '*...For it is a proposition well known to geometers, that the velocity of a pendulous body in the lowest point is as the chord of the arc which it has described in its descent*', after a full swing the pendulum will return to point V so that RV represents the full retardation. As one complete oscillation is made up of four quarter oscillations, and the four retardations are increasingly smaller, one has to determine how much the first retardation contributes to the full RV. The easiest solution is to say that all contribute the same amount $(1/4)RV$. However, if one wants to minimize his error, one can devise a trick: there must exist a point S below R so that the momentum the bob loses upon reaching A will be exactly $RS = RA - SA = (1/4)RV$. If one is able to find this S, then one can be sure that a bob starting from that point will lose momentum which corresponds exactly to a

retardation $(1/4)RV$. Newton knows that S should actually be placed a little further down, but not too much. So he chooses a point T in order to constrain how far down he can place S by making $ST = 1/4RV$. He then places ST halfway between the observed values of R and V (see Fig. 1). Newton's original passage reads (Newton 1687, p. 20):

> *Bring the body A to any point R of the arc EAF, and (withdrawing the body B) let it go from thence, and after one oscillation suppose it returns to a point V: then RV will be the retardation arising from the resistance of the air. Of this RV let ST be the fourth part, situated in the middle, namely so that*

$$RS = TV \tag{3}$$

> *and*

$$RS : TV = 3 : 2 \tag{4}$$

> *then will ST represent very nearly the retardation during the descent from S to A.*

There does not seem to be any particular good reason for choosing this point other than the fact that he knows that the quarter retardations are not equal and the first one is largest. So instead of choosing

$$S = R - \frac{1}{4}RV \tag{5}$$

he chooses

$$S = R - \left(\frac{1}{4}RV + \frac{1}{8}RV\right) \tag{6}$$

He then proceeds, considering what happens at the other side, where the bobs ascend (Newton 1687, p. 20-21):

> *Restore body B to its place: and supposing the body A to be let fall from the point S, the velocity thereof in the place of reflection [i.e. collision] A, without sensible errors, will be the same as if it had descended in vacuum from the point T … After reflection, suppose the body A comes to the place s and the body B to the place k. Withdraw the body B, and find the place v, from which, if the body A, being let go, should after one oscillation return to the place r, st may be a fourth part part of rv, so placed in the middle thereof as to leave rs equal to tv, and let the chord of the arc tA represent the velocity which the body A had in the place A immediately after reflection. For t will be the true and correct place to which the body A should have ascended, if the resistance of air had been taken off. In the same way we are to correct the place k to which the body B ascends, by finding the place l to which it would have ascended in vacuum. And thus everything may be subjected to experiment, in the same manner as if we were really placed in vacuum.*

So, by using similar procedures for the ascent of the bobs, he is able to correct for their momentum. His conclusion:

> *Thus trying the thing with pendulums of 10 feet, in unequal as well as equal bodies, and making bodies concur after a descent through large spaces, as of 8, 12 or 16 feet, I found always, without an error of 3 inches, that when bodies concurred together directly, equal changes towards the contrary parts were produced in their motions, and, of consequence, that the action and reaction were always equal.*

This are the last words of Newton that Diderot transcribes in his Memoir. The he goes on to determine, in a rigorous way, the location of S under the assumption of an air resistance force linear in v.

4.2 Diderot's Solution: Lettre sur la Résistance de l'Air

Since Diderot's article aims at explaining how Newton got his S while at the same time changing the hypothesis as to what regards the kind of drag one uses, his Memoir is didactically organized in three mains parts: two Propositions and the *Éclaircissements* (Clarifications). The Propositions deal with ways of calculating the retardation in the case of a quadratic resistance force. The presentation, which might at first seem quite awkward to the reader, follows closely Newton's logic except that Diderot uses a different hypothesis and makes extensive use of differential tools. The *Éclaircissements* are the place where Diderot actually solves for Newton's S. Rather surprising is the fact that the *Éclaircissements* can be read quite independently, since Diderot makes no direct use of his results from Proposition I and II. So one may rightfully ask the reason why he goes at pains to do all the calculations which he does not use at the end. In the authors opinion this was part of his strategy: other than simply offering the solution to a problem posed, Diderot shows in the *Éclaircissements* that Newton's answer is an approximation to the full solution which would follow as a direct consequence of the *methodological* approach that he, Diderot, developed. If his reasoning, mathematically formulated, allowed him to go beyond Newton, then his solutions of Propositions I and II must be correct. The emphasis is on the *method*, not on the *solution*

Thus, he starts each proposition in the form of a homework, a *Probléme* that he poses: to find the velocity v of a bob for an arbitrary point M along the trajectory given that besides the weight, the bob is also acted upon by a retarding force proportional to v^2. Proposition I deals with the bob's way from B (to the left of the vertical OA) as it moves down to A, the lowest point of the trajectory (see Fig. 2). Proposition II deals with the movement of the bob initially at A as it moves upwards the right after being given an initial velocity h. The separation of the question into two separate ones is due to the fact that Newton discusses each quarter cycle independently. This is natural in the context of Newton's commentaries: since Newton was interested in the collision of two bobs, the descending bob will execute a quarter of a cycle before colliding. After Diderot gives a *Solution* to a *Probléme*, he writes down a few extra corollaries, which are either straightforward consequences of his main solution or approximations that one gets when considering small angles of oscillation.
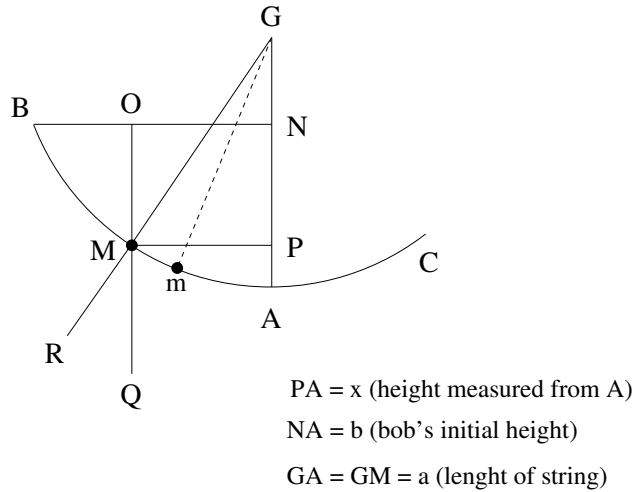
Proposition I: *Let a pendulum A which describes an arc BA in air be attached to the string GM fixed at G. One asks for the velocity of this pendulum for any point M, assuming that it starts falling from point B.* (Diderot 1748, p. 321).

Before we discuss Diderot's solution, his choice of variable requires some explaining: instead of using θ, the displacement angle, as one would normally do nowadays, he prefers to think in terms of the height x of the bob relative to the lowest point A of the trajectory. There is a reason for this: in the absence of damping, by conservation of energy we know that the change in kinetic energy of the bob is equal to change in potential energy. This allows one to directly find the velocity at a given height x_1 by giving the difference in height $x_0 - x_1$ through which the bob of mass m fell, that is

$$\frac{1}{2}mv_1{}^2 - \frac{1}{2}mv_0{}^2 = mg(x_0 - x_1) \longrightarrow v_1{}^2 = v_0{}^2 + 2g(x_0 - x_1). \qquad (7)$$

This is Torricelli's equation for a body with acceleration g. Even though this result does not hold in the presence of damping, one may still use it as a first approximation to the real velocity, as Diderot eventually did.

The height from which the bob starts is the orthogonal projection of point B on line OA, and this Diderot calls $b = x_0 = x(t = 0)$ (see Fig. 2). The length of his

PA = x (height measured from A)

NA = b (bob's initial height)

GA = GM = a (lenght of string)

Fig. 2. The pendulum in Diderot's work: M represents an arbitrary point of the trajectory of a bob dropping from the initial position B. The position m is infinitesimally close to M. Diderot expresses the position of the bob in terms of the height x of point M, measured relative to the lowest point of the trajectory, that is, the segment AP (per definition height at A is $x = 0$). The initial height is $b = NA$. The length of the pendulum is $a = GA = GM$.

string is a (usually called l in modern texts). He assumes that the force due to air resistance is given by

$$F(v) = \gamma \, v^2 \tag{8}$$

which he writes as

$$F(v) = \frac{f}{g^2} \, v^2 \tag{9}$$

The modern reader might find this a bit confusing but this equation comes from the fact that Diderot assumes that for a given known velocity g (not to be confused with the acceleration of gravity) the force has a known value of f. So, from (8) one has

$$F(g) = f = \gamma \, g^2 \longrightarrow \gamma = \frac{f}{g^2} \tag{10}$$

The factor f/g^2 is carried along through the whole text. The determination of this term is no simple experimental task and Diderot lacked access to scientific apparatus. So, even if conceptually correct, the use of the factor f/g^2 might have served him the purpose of convincing his readers that the problem was real, not just a toy model. For the sake of a more compact notation we will keep the parameter γ where Diderot uses f/g^2 and think of its determination as it is normally done in a laboratory experiment: by fitting the amplitude as it decays with time or by measuring the drag in a wind tunnel.

Diderot's approach consists in finding a relation between dv, the increment in velocity, and the difference in height dx associated to the fall. The equation of motion for a bob of weight p acted upon by a force of the type Eq. (8) is

$$m\frac{dv}{dt} = p\sin\theta - \gamma\, v^2 \qquad (11)$$

To write it in terms of dx Diderot needs to find a way to relate this to dt. He begin by noticing that

$$dt = \frac{ds}{v} \qquad (12)$$

where ds (Diderot's arc Mm) is the length the bob traverses along the arc during the time interval dt. So, replacing dt by this expression he gets

$$m\, dv = (p\sin\theta - \gamma\, v^2) \times \frac{ds}{v} \qquad (13)$$

or

$$v\, dv = (p\sin\theta - \gamma\, v^2) \times \frac{ds}{m} \qquad (14)$$

In Diderot's original work the mass m of the pendulum does not appear. This could be a lapse, not a conceptual mistake, or the fact that Diderot took $m = 1$ without mentioning it. For the sake of completeness we will keep the mass m in the equations that follow. To go over now to Diderot's x one has to first remember that an infinitesimal arc ds is related to the infinitesimal angular displacement $d\theta$ via $ds = a\, d\theta$. The relation between θ and x can be easily inferred from Fig. 3 and some basic trigonometry:

$$\sin\theta = \pm\frac{\sqrt{2ax - x^2}}{a} \qquad (15)$$

The $\pm$ sign comes from the fact that the expression on the right-hand side is always positive but $\sin\theta$ can be either positive or negative depending on the which side the bob is. In other words, for a bob moving from left to right one always has $d\theta > 0$ while on the journey down $dx < 0$ while on the way up $dx > 0$. This is what Diderot means when he says (Diderot 1748, p. 323):

> In this equation I substitute the little arc Mm [ds] by its value $-\frac{a}{\sqrt{a^2 - b^2}}\, dx$, with a minus sign, because as the pendulum goes down the velocity increases while x becomes smaller.
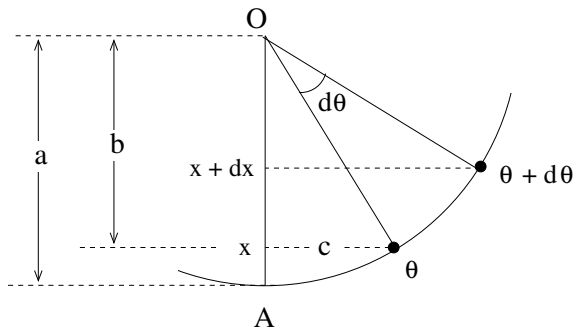
Differentiating both sides of Eq. (15) one gets

$$\cos\theta\, d\theta = \pm\frac{a - x}{a\sqrt{a^2 - b^2}}\, dx \qquad (16)$$

or remembering that $\cos\theta = \frac{a-x}{a}$ this can be recast as

$$d\theta = \pm\frac{dx}{\sqrt{a^2 - b^2}} \qquad (17)$$

from which one gets

$$v\, dv = p\,(-dx) - \gamma\, v^2 \times \frac{a(-dx)}{\sqrt{2ax - x^2}} \qquad (18)$$

Fig. 3. The relation between θ and x, which Diderot uses as variable in his memoir. The angle θ is at the vertex of a right triangle of sides a, b, and c. From these lengths and a few trigonometric identities one can find the relation between θ and x.

which is how Diderot writes (14). Integrating both sides of this equation one ends up with

$$\frac{v^2}{2} = -\frac{p}{m} \int_b^x dx' + \int_b^x \frac{\gamma}{m} \frac{v^2 \, a \, dx'}{\sqrt{2ax - x^2}} \tag{19}$$

or

$$\frac{v^2}{2} = \frac{p}{m}(b - x) + \int_b^x \frac{\gamma}{m} \frac{v^2 \, a \, dx'}{\sqrt{2ax - x^2}} \tag{20}$$

Another point to note is that Diderot never writes explicitly the upper and lower limits of integration but does it by explicit comments in the text.

Diderot thus ends up with an integral equation for v, which he cannot solve. He notes however that, in the absence of air, the speed of a pendulum falling from rest from B to M, that is, from a height b to a height x is simply

$$\frac{mv^2}{2} = p(b - x) \tag{21}$$

which follows from conservation of energy. In order to handle Eq. (20) Diderot uses the following argument: given that the drag is much smaller than the weight of the bob, one may assume that 'v^2 *différera très–peu de 2 p b − 2 p x*' [notice there is a mass m missing in Diderot's calculation]. One may, therefore, substitute v^2 inside the integral by its approximate value $(2 \, p \, b - 2 \, p \, x)/m$ to finally write

$$v^2 = 2 \frac{p}{m}(b - x) + 2 \int_b^x \frac{\gamma}{m^2} \frac{(2pb - 2px') \, a \, dx'}{\sqrt{2ax' - x'^2}} \tag{22}$$

What Diderot does, in modern parlance, is a first-order approximation, that is, to substitute for v^2 in the integral the value it would have in vacuum and thus obtain a correction. A zeroth-order approximation would be to assume v^2 in air to be the same as in vacuum.

In the solution of (22) one can nicely see how Diderot thought in terms of geometrical images, but did not shun the analytical approach: he mixes geometrical ideas with analytical ones find a solution to his integral.

The first integral to be solved is

$$2 \int_b^x \frac{\gamma}{m^2} \frac{2pb\, a\, dx'}{\sqrt{2ax' - x'^2}} = \frac{4\, p\, b\, \gamma}{m^2} \int_b^x \frac{a\, dx'}{\sqrt{2ax' - x'^2}} \tag{23}$$

But Diderot knows, without bothering to say, that the integrand is just the infinitesimal arc Mm, so the integral is nothing but the arc measure from point B to point M, that is, his BM. So he writes his answer as

$$\int \frac{b\, a\, dx'}{\sqrt{2ax' - x'^2}} = -b \times\ BM \tag{24}$$

where the $-$ sign comes from his convention for the sign of dx. The remaining part of (22) is a bit harder. Diderot writes down

$$\int \frac{-a\, x'\, dx'}{\sqrt{2ax' - x'^2}} = \int \frac{(a^2 - a\, x')\, dx'}{\sqrt{2ax' - x'^2}} - \int \frac{a^2\, dx'}{\sqrt{2ax' - x'^2}} \tag{25}$$

This rewriting of the equation, by adding and subtracting the same term, is easily explained. Diderot knows that

$$a \frac{d}{dx}(\sqrt{2ax - x^2}) = \frac{a^2 - ax}{\sqrt{2ax - x^2}} \tag{26}$$

and he can thus write (25) as

$$\int \frac{-a\, x'\, dx'}{\sqrt{2ax' - x'^2}} = a \int \frac{d}{dx'} \sqrt{2ax' - x'^2}\, dx' - a \int \frac{a\, dx'}{\sqrt{2ax' - x'^2}} \tag{27}$$

The solution of the first integral on the right hand side is trivial. Moreover, and here his geometrical intuition comes to his help, the integrand $\sqrt{2ax - x^2} = \sqrt{a^2 - (a - x)^2}$ is just the distance from point M to the vertical axis OA, that is, the straight line MP. With can then finally bring all these results into one equation and write [12]

$$mv^2 = 2\, p\, (b - x) - \frac{4\, \gamma}{m}\, p\, b \times\ BM - \frac{4\, \gamma}{m}\, p\, a \times\ (BO - BM) \tag{28}$$

which is the solution of his original question expressed in terms of the arc BM and the distance BO. In Diderot's variable x this would read:

$$mv^2 = 2\, p\, (b - x) + \frac{4\, \gamma}{m}\, p\, a\, (a - b) \left[\cos^{-1}(1 - b/a) - \cos^{-1}(1 - x/a) \right]$$
$$- \frac{4\, \gamma}{m}\, p\, a \times (\sqrt{2ab - b^2} - \sqrt{2ax - x^2}) \tag{29}$$

There is however a certain charm (and economy) in Diderot's original notation, because his equation allows one to come up with a nice geometrical interpretation of velocity correction directly in terms of the distances traversed by the pendulum: BM represents the distance the bob travels along the circular arc and BO measures how

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[12] Recalling that Diderot uses f/g^2 instead of γ, there is a misprint in the original: a factor of g^2 is missing in the denominator of the $(B0 - BM)$ term.

far it moves to the right, from B to M. This allows one also to study some interesting limits, which Diderot does in corollaries I, II and III. Corollary I is just the expression above calculated for $x = 0$, that is, what the velocity looks like when the bob reaches A. Corollary II is a rather trivial observation (but important for Diderot's subsequent discussions) and follows from a rewriting of the equation above: that the velocity obtained is the same velocity of pendulum that falls without air resistance from a starting point below point B. He puts the problem always in terms of comparisons between fall in vacuum as opposed to fall in viscous medium in order to prepare the reader for his discussion of Newton.

Corollary III follows when considering what would happen if one had a small initial amplitude. In this case $BM \approx BO$. With $x = 0$ Diderot's expression becomes

$$mv^2 = 2\,pg\,b\left(1 - \frac{2\,\gamma}{m}BM\right) \qquad (30)$$

which is Diderot's correction to Torricelli. With these 3 corollaries Diderot moves over the the second part of his problem: how to determine the velocity of the bob on the way up, for an arbitrary point M, given an initial velocity at A equal to $v_A^{(0)}$ (Diderot calls this initial velocity h). The part devoted to the second proposition is longer, not because the problem is more difficult – what he has to do now is basically to revert the sign of dx in the equation he already had and add an initial velocity h – but because in this section he derives the result that retardation goes as $(\text{arc})^2$ and not linear in the arc, as Newton assumed.

Proposition II. *'Suppose that a pendulum A, placed initially at the vertical GA, is given an impulse or velocity h along the horizontal AR. One wants to know its velocity for an arbitrary point M.'* (Diderot 1748, p. 324).
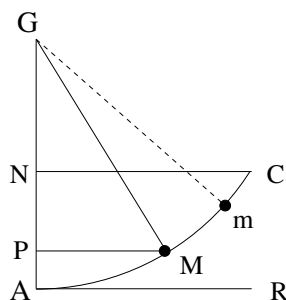
Fig. 4. The figure used by Diderot in his study of the movement of the bob upwards. As always, Mm represents an infinitesimal displacement for Diderot.

Diderot repeats the same steps as before but making sure that in the new equation

the signs of dv and dx are opposite, as the velocity decreases as the bob moves upward (see Fig. 4). Once again he uses Torricelli's equation

$$v^2 = (v_A^{(0)})^2 + 2\,a\,(x_{final} - x_0) = h^2 - 2\frac{p}{m}x \tag{31}$$

for an initial velocity $v_A^{(0)} = h$ and acceleration $a = p/m$ to solve the integral in approximate form. As his calculations are basically the same, we will not repeat them. He arrives at the following answer

$$mv^2 = mh^2 - 2\,p\,x - \frac{2\,\gamma}{m}\,h^2 \times\ AM - -\frac{4\,\gamma}{m}\,p\,a \times\ (AM - MP) \tag{32}$$

He now takes a different path. Instead of leaving the equation as it is, he substitutes h^2 by the maximum height AN the pendulum would reach in vacuum. This can be easily done since the highest point is where $v = 0$. So, if one takes the expression above for $v = 0$ and $\gamma = 0$ he finds

$$m\,h^2 = 2\,p \times AN \tag{33}$$

Substituing this value back into his answer and noticing that $(AN - x) = NP$, he writes

$$mv^2 = 2\,p \times\ NP - \frac{4\,\gamma\,p}{m}\ \times\ AM \times AN + \frac{4\,\gamma\,p}{m}\ \times\ a \times\ (AM - MP) \tag{34}$$

This expression is the starting point for his corollary I of proposition II, namely, to find the highest point reached by the pendulum in the presence of drag. This can be easily obtained by setting $v = 0$ in the expression above and finding the respective x_{max}. He calls this point c (see Fig. 5) but, in a rather confusing way, he gives his answer in terms of the difference between the highest point in vacuum AN and the highest point in the presence of air An. So, following his line of thought, $x_{max}^{(vac)} - x_{max}^{(air)} = AN - An = Nn$ and he finally writes that there will be a point c where the pendulum will revert its motion. From this follows

$$Nn = 2\frac{\gamma}{m} \times AN \times Ac + \frac{2\gamma\,a}{m} \times (nc - Ac) \tag{35}$$

In corollary II Diderot gives an approximation for the above expression in terms of the results one would get in vacuum. He notes that the arc Ac differs very little from the vacuum value AC and the same can be said about the height nc, which differs little from NC. So, he just rewrites the result above replacing nc by NC and Ac by AC to get

$$Nn = 2\frac{\gamma}{m} \times AN \times AC + \frac{2\gamma\,a}{m} \times (NC - AC) \tag{36}$$

Corollary III consists in recasting the expression above when the oscillation amplitude is small, in which case $AC \approx NC$. This amounts to making the last term on the right-hand side of the previous expression equal to zero and keeping only the first term [13].

From these considerations Diderot now calculates the maximum height $A\nu$ the bob reaches when being let go from B (see Fig. 5). Finding ν is equivalent to finding point k along the trajectory to which it corresponds. With k one may then calculate Ck, which is equivalent to Newton's RV. To find this point Diderot uses the following

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[13] In the original memoir there is a misprint, when Diderot says that AC should be 'almost equal' to AN. One should substitute AN by NC.

argument: C represents the point in the trajectory opposite to B, the starting position of the bob. As the bob is acted upon by some drag, it will not reach C but a certain k below C. So, Ck is the different in path between a bob with and without drag. However, a bob falling from B with air resistance is equivalent to a bob falling from an point below B without air resistance. This point is the one opposite to n, between C and k, in Fig. (4).

He now sums up his preceding results: from Proposition I he found the velocity v the bob has when reaching the lowest point A. It is the same velocity the bob would have if it fell from a point below B without air resistance. Calling now this velocity h, he uses it as a starting velocity for the ascending bob.

So, from Cor. II of Prop. I one can say that a bob falling from height $b = AN$ with air resistance is the same as falling from height $An < AN$ without air resistance:

$$An = b - 2\frac{\gamma}{m} \times b \times BA - \frac{2\gamma}{m} \times a \times (BN - BA) \tag{37}$$

Consequently, from Cor. II of Prop. II it follows that the bob will not go up to the opposite of point because of air resistance, but to a point k (of height $A\nu$) slightly before c (of height An).

$$A\nu = An - 2\frac{\gamma}{m} \times An \times AC + \frac{2\gamma}{m} \times a \times (nc - Ac) \tag{38}$$

Substituting in this expression the value of An just found, and using the small angle condition such that $nc \approx BN$ and $Ac \approx BA$ one ends up with

$$A\nu = b - \frac{4\gamma}{m} \times b \times BA + \frac{4\gamma}{m} \times a \times (BN - BA) \tag{39}$$

which is the content of Corollary III. Corollary IV is deduced from the fact that, when angles are small, $BN \approx BA$ and the expression above reduces to

$$A\nu = b - \frac{4\gamma}{m} \times b \times BA \tag{40}$$

The most important section of the Memoir, at least as to what regards Diderot's disagreement with Newton, is Corollary V. He has to explicitly give an expression for Ck. From (39) one may get, for small angles ($BA \approx BN$) the simplified expression

$$A\nu = b\left(1 - \frac{4\gamma}{m} \times BA\right) = AN\left(1 - \frac{4\gamma}{m} \times BA\right) \tag{41}$$

Now comes one section of Diderot's memoir which does justice to his style: what seems trivial is not worth explaining in more detail. He says that for small angles, the arc AC is to Ak as the root of AN is to the root of $A\nu$. He adds: *... since in the circle, the chords are among them as the roots of the abscissae; or the arcs can be replaced here by the chords*. Diderot writes this as

$$\frac{Ck}{AC} = \frac{\sqrt{AN} - \sqrt{A\nu}}{\sqrt{AN}} \tag{42}$$

To see how one can get this, consider Fig. (6). The chord s of Fig. (6) can be written in terms of the radius a and the angle θ by means of the cosine law

$$s^2 = 2\,a^2 - 2\,a^2\,\cos\theta \tag{43}$$
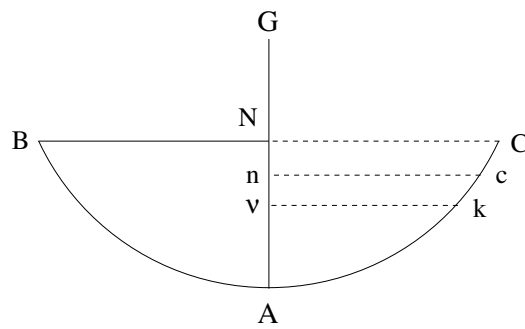
Fig. 5. The figure in Diderot's memoir depicting the highest point the bob reaches in the presence of air resistance. This is point ν on the vertical axis, which corresponds to point k of the trajectory.

s = chord
a = radius
b = initial height of bob
l = arc
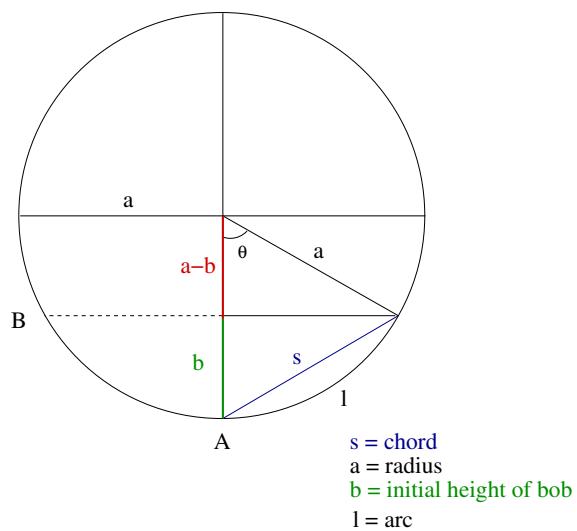
Fig. 6. The geometric construction to deduce Eq. (42) of Diderot's memoir.

Since $\cos\theta = (a - b)/a$ one may substitute this in the expression above to get

$$s^2 = 2\,a^2 - 2\,a^2 \times \frac{a - b}{a} \longrightarrow s = \sqrt{2ab} \tag{44}$$

Given that Diderot is considering small angles, one can approximate arcs by chords and thus write, in an approximate way

$$l \sim \sqrt{2ab} \tag{45}$$

From this one gets

$$\frac{Ck}{AC} = \frac{AC - Ak}{AC} = \frac{\sqrt{2a \times \ AN} - \sqrt{2a \times \ A\nu}}{\sqrt{2a \times \ AN}} \qquad (46)$$

which is the same as Eq. (42). By using the value of $A\nu$ found in Eq. (40) and a series expansion for the square root

$$\sqrt{1 + x} \sim 1 + \frac{1}{2}x \qquad for \qquad x \ll 1 \qquad (47)$$

Diderot arrives, after some straightforward algebra, at the result

$$Ck = 2 \times \frac{\gamma}{m} \times (AB)^2 \qquad (48)$$

which he expresses in terms not of AC, but of AB, since these arcs have the same length. Thus difference in the arc due to air resistance is proportional to the square of the trajectory of the bob on its way down. This is Diderot's main result and the point of the memoir where he confronts Newton.

If we compare his solution Eq. (48) with Eq. (114) obtained via a Lindstedt-Poincaré Method, we can write the latter in Diderot's notation:

$$Ck = \frac{4}{3} \times \frac{\gamma}{m} \times (AB)^2. \qquad (49)$$

where one can clearly see that prefactors of the two solutions differ. If we want to understand the reason why he did not get it right one may look back at Eqs. (35), (37), (38) and Fig. (4). The first of these is an equation for the point Diderot denotes by n along the vertical OA. This is the highest point the bob would reach had it started with velocity h at the bottom. The expressions on both sides of this equation involve the unknown n, as it is hidden in the definition of the arcs nc and Ac. To make this point more clear, we can rewrite (35) in terms of the variables θ_0 (the angle of point B) and θ_1, the maximum value of θ on the bob's way up (to which height n is associated). One obtains

$$\cos\theta_1 - 2\frac{\gamma}{m}a \sin\theta_1 + 2\frac{\gamma}{m}a\,\theta_1 = \cos\theta_0 + 2\frac{\gamma}{m}b\,\theta_0 \qquad (50)$$

This is a transcendental equation for the unknown θ_1. Instead of solving for θ_1 (or n, which is the same), he approximates n by N and c by C on the right-hand side of (35). This is the same as replacing θ_1 by θ_0 in those terms

$$\cos\theta_1 - 2\frac{\gamma}{m}a \sin\theta_0 + 2\frac{\gamma}{m}a\,\theta_0 = \cos\theta_0 + 2\frac{\gamma}{m}b\,\theta_0 \qquad (51)$$

to get

$$\cos\theta_1 = \cos\theta_0 + 2\frac{\gamma}{m}(b - a)\,\theta_0 + 2\frac{\gamma}{m}a \qquad (52)$$

He then subtracts (35) from AN to get (37). He then proceeds to (38), keeping n and c, but then approximate them again to get (40). To conclude, Diderot's approach is to avoid solving the transcendental equation, by approximating his unknowns c and n by their values C and N in vacuum. By doing this he loses on the way all important terms in the approximation which sum up to give him the correct prefactor, while still getting the right functional dependence on AB.

There follows three corollaries (VI, VII and VIII) which refer to ways of determining position k (or ν) in a back-of-the-envelope kind of calculation. They are

straightforward consequences of the result he derived in Corollary V. We reproduce
them here for the sake of completeness.

Corollary VI. If one knows the arc ABC that a pendulum traverses when let go from
B, one can easily find arc bAk which is the trajectory when let go from b. One just
needs to find Ak, which one can get from

$$\frac{BA - AC}{bA - Ak} = \frac{BA^2}{bA^2} \tag{53}$$

Corollary VII. Thus, if a pendulum falls through BA in air, one can find its velocity
in at point A by dividing the $N\nu$ into two equal segments marked by point n. This is
so since this velocity, according to Corollary III of Proposition I, is almost the same
as that obtained by a pendulum in vacuum from point $b - (2\gamma/m) \times BA = b - N/2$.

Corollary VIII. One has

$$\frac{AC^2}{Ac^2} = \frac{AN}{An} \tag{54}$$

that is

$$\frac{AC}{AC^2 - 2Cc \times AC} = \frac{AN}{AN - Nn} \tag{55}$$

from which follows

$$Nn = \frac{2Cc \times AC \times AN}{AC^2} = \frac{2Cc \times AN}{AC} \tag{56}$$

For the same reason one has

$$N\nu = \frac{2Ck \times AN}{AC} \tag{57}$$

and thus

$$\frac{Ck}{Cc} = \frac{N\nu}{Nn} \tag{58}$$

Thus c is the point in the middle of arc Ck. This means that, instead of dividing $N\nu$
into two equal parts, one may divide Ck into two equal parts in order to obtain the
arc Ac that body A will have traversed in vacuum.

   With these results he shows that if one consider a resistance force quadratic in the
velocity, one indeed gets a retardation which is proportional to the square of the arc
AB. He further justifies his results with a bit dimensional analysis, before he moves
on to his *Eclaircissements*. His idea is the following (Diderot 1748, p. 326):

> *If pendulum A is a small sphere, the resistance f, all other things being equal,
> is inversely proportional to the diameter of this sphere and its density; since
> the resistance caused by air on two spheres of different diameters goes as the
> surface or the square of the diameter; and this resistance has to be divided by
> the mass, that is like the density multiplied by the third power of the diameter.
> Thus the arc Ck, all other things being equal, is like AB^2 divided by the product
> of the diameter of the sphere and its density.*

How is this to be understood? Diderot is correct when he affirms that the resistance
goes as the surface, as we now in hindsight that it depends on the Reynolds number

Eq. (97). But when he affirms that 'this resistance has to be divided by the mass' it would mean, according to his reasoning, that

$$F_R \sim \frac{diameter^2}{mass} = \frac{diameter^2}{density \ \times \ diameter^3} = \frac{1}{density \ \times \ diameter} \qquad (59)$$

Diderot is not too rigorous with his wording, since from the sentence above the 'resistance' f cannot be the same f he is using to mean 'resistance of air' throughout the text. He probably has as 'acceleration' in mind. This is so since the drag force depends only on the geometry of the bob. If they have the same diameters, the drag is the same. However, the equation of motion in the two cases, given that they have different masses m_1 and m_2 is

$$m_1 \, g \sin\theta - F_R = m_1 \, a_\theta \qquad m_2 \, g \sin\theta - F_R = m_2 \, a_\theta \qquad (60)$$

From which it trivially follows that the accelerations a_θ along the tangential of the arc are different in the two different cases,

$$a_{1,\theta} = g \sin\theta - \frac{F_R}{m_1} \quad and \quad a_{2,\theta} = g \sin\theta - \frac{F_R}{m_2} \qquad (61)$$

Moreover, we know that the expansion of Lindstedt-Poincaré is an approximation valid for small values of $\epsilon = (\gamma \, l/m)$ which, for a fixed string length l, takes exactly into account the ratio of the damping parameter and the bob's mass (see discussion in Section 5 below). His intuition got him on the right track.

4.3 Diderot's Éclaircissements of Newton

As discussed in Section 4.1, Newton explained the difference between experimental data and theoretical values in his pendulum experiment as a consequence of air resistance. He gave an approximate value for S (see Eq. 6) which Diderot now calculates under the assumption of a linear drag. To better understand Diderot's solution, we reproduce Fig. 7 that Diderot uses in his Memoir while explaining the solution.

Problem: find the location of point S such that a bob falling from it to point A will have a retardation which is exactly equal to $1/4$ of a full cycle retardation RV.

In his Memoir Diderot chooses the arcs such that $RA = 1$, $RV = 4b$, and $SA = x$. He sets out to find x. The choice of variables is an indication that he also studied the *Principia* from the annotated editions of Le Seur and Jacquier, since when referring to the same passage of Newton that Diderot is addressing they say (Le Seur and Jacquier 1739, p.37):

> *Bring body A to any point R along the arc EAF and let it fall from there. If the resistance of the medium is absent, it will reach the same height M to which it was lifted and should return to R. But when, after the first oscillation composed of exit and return, it returns to point V (according to the hypothesis), the arc RV will represent the retardation of a double ascent and descent [caused by the] medium; thus one should take the retardation due to the medium in one whole descent as the fourth part of the total retardation, that is the fourth part of arc RV, provided it did not descend neither from the highest point R nor from the lowest V to begin with: for the retardation will be larger for the larger arc than the smaller one, since as the pendulum describes ever smaller oscillations,*
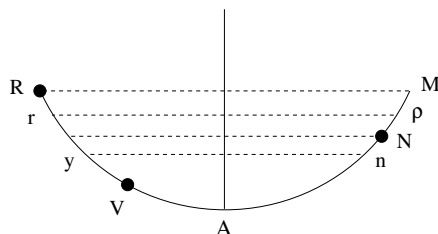
Fig. 7. The figure Diderot uses to explain his determination of the point S in Newton's commentaries. In the figure in the original Memoir there is a misprint: the letter y should be opposite to n and not N. The pendulum is let loose from R, reaches N on the opposite side and return to V. For the sake of clarity, these points are marked here by black dots. RV is the retardation of a full cycle.

the retardation of each single arc will be unequal, and the retardation of the descent by RA will be bigger than the fourth part of RV, and the retardation of the last ascent AV will be smaller than the fourth part of the total retardation RV. With a similar calculation Newton determined a point S such that the retardation in descending through SA should be [exactly] the fourth part of of the total retardation RV. Let arc RA be 1, arc RV be $4b$ and the arc sought SA be x; since the retardation is proportional to the arc [traversed], the arc SA (x) is to the arc RA (1), as the retardation of the arc SA, defined as b, the fourth part of the whole RV, is to the retardation of the first arc RA, that is b/x. The successive delays to be found, the second, the third and the fourth follow the same ratio. The arc of the second is equal to RA, allowing for a retardation of b/x. The third arc is equal to the second, allowing for the same retardation, and so on, but all these delays sum up to give the whole RV, or $4b$. Hence [we obtain] the equation from which we get the value of the arc SA, or x, [which] by means of an approximation [turns out] to be equal to $1\ 3/2\,b$ [that is $RA\ 3/2\,b$]. So taking RS equal to the fourth part of the arc RV with its half, the retardation of the arc SA will be equal to ST, the fourth part of the total retardation RV, and therefore a body dropped from point S will have the same speed at A, without significant error, as it would have if it had fallen in vacuum from T.

This is the origin of Diderot's naming of arcs and takes him one step further, indicating how the calculation should be done. It does not spare him however the work of actually finding S. For the sake of completeness and further comparison of lengths, we will keep RA arbitrary while maintaining Diderot's value for RV and SA, that is $4b$ and x. We closely follow Diderot's ideas up to his solution.

If a body falls from A, it will return to V. Each quarter cycle contributes a retardation r_i such that

$$r_1 + r_2 + r_3 + r_4 = RV \tag{62}$$

We know that the retardations are not equal, that is $r_i \neq r_j$ for $i \neq j$. As an approximation, we can think of a point below R – call this point S – such that a bob falling from it until A will show a quarter-cycle retardation exactly equal to $(1/4)RV = b$. This must be so since we know that a body falling from R will have a retardation $r_1^{(R)} > b$, so to have a retardation smaller requires S to be further down the track. Newton's hypothesis on the retardation is that is proportional to RA,

$$Rr = \alpha RA \tag{63}$$

where α is some constant. It is important to recall that Newton's argument is based on the idea that a bob falling with resistance from R is the same as falling without resistance from a lower point r. So if one determines the velocity v_A at A in air, one can just use some reverse engineering and determine which r would give that same v_A in vacuum. This is trivial, since then one may just use conservation of mechanical energy to find r. Following this idea, Diderot assumes that a bob falling from S would be the same as a bob falling in vacuum from r' and, given Newton's assumption, one would have in place of the equation above

$$Sr' = \alpha SA \tag{64}$$

where Sr' is the retardation when falling from S. But the problem is to find S for which this retardation is exactly b. So, by eliminating α in the equations above one gets

$$\frac{Rr}{RA} = \frac{Sr'}{SA} \longrightarrow Rr = \frac{Sr'}{SA}\, RA \tag{65}$$

Since $Sr' = b$ and $SA = x$ this reduces to

$$Rr = RA - rA = \frac{b}{x}\, RA \tag{66}$$

Now, the arc described on the first ascent would be $A\rho = Ar = (1 - b/x)\, RA$ but due to air resistance the bob does not reach ρ but a lower point N such that

$$\rho N = \alpha A\rho = \alpha Ar = \alpha\left(1 - \frac{b}{x}\right) RA = \frac{b}{x}\left(1 - \frac{b}{x}\right) RA \tag{67}$$

So the actual arc the bob describes is

$$AN = A\rho - \rho N = \left(1 - \frac{b}{x}\right) RA - \frac{b}{x}\left(1 - \frac{b}{x}\right) RA = \left(1 - \frac{b}{x}\right)^2 RA \tag{68}$$

So, by following this kind of argument one can determine all four quarter cycle retardations. They are

$$Rr = \frac{b}{x}RA$$

$$\rho N = \frac{b}{x}\left(1 - \frac{b}{x}\right) RA$$

$$Nn = \frac{b}{x}\left(1 - \frac{b}{x}\right)^2 RA$$

$$Vy = \frac{b}{x}\left(1 - \frac{b}{x}\right)^3 RA \tag{69}$$

The sum of all these retardations should be $4b$, that is

$$RA\left[\frac{b}{x} + \frac{b}{x}\left(1 - \frac{b}{x}\right) + \frac{b}{x}\left(1 - \frac{b}{x}\right)^3 + \frac{b}{x}\left(1 - \frac{b}{x}\right)^3\right] = 4b \qquad (70)$$

This leads to a quartic equation in the unknown x

$$\frac{1}{RA}x^4 - x^3 + \frac{3b}{2}x^2 - b^2 x + \frac{b^3}{4} = 0 \qquad (71)$$

Before solving this equation, Diderot considers the limiting case where $b \ll 1$, in which case one may neglect the last two terms on the right hand side and write

$$\frac{1}{RA}x^4 - x^3 + \frac{3b}{2}x^2 = 0 \longrightarrow x^2 - RA\,x + \frac{3b}{2}\,RA = 0 \qquad (72)$$

This equation has two solutions, namely

$$x_{+,-} = \frac{RA}{2} \pm \frac{RA}{2}\sqrt{1 - \frac{6b}{RA}} \qquad (73)$$

If one further considers an approximation to the square root given by Eq. (47) these solutions reduce to

$$x_+ = RA - \frac{3}{2}b$$

$$x_- = \frac{3}{2}b \qquad (74)$$

Solution x_- is not physically acceptable, since it would imply that S is close to A. Solution x_+ can be written as

$$x_+ = RA - \left(b + \frac{1}{2}b\right) \qquad (75)$$

If one recall that b is what Newton called $(1/4)RV$, one can write

$$x_+ = RA - \left(\frac{1}{4}RV + \frac{1}{8}RV\right) \qquad (76)$$

which is the same as Eq. (6). So, Diderot shows that Newton's placement of S can be recovered in the limit where b is taken as being very small. But Diderot goes a bit further, by solving exactly the quartic. He does this by expanding the exponents in (70) and rewriting it as

$$\left(4\frac{b}{x} - 6\frac{b^2}{x^2} + 4\frac{b^3}{x^3} - 6\frac{b^4}{x^4}\right) = \frac{4b}{RA} \qquad (77)$$

He then notices that

$$1 - \left(4\frac{b}{x} - 6\frac{b^2}{x^2} + 4\frac{b^3}{x^3} - 6\frac{b^4}{x^4}\right) = \left(1 - \frac{b}{x}\right)^4 \qquad (78)$$

and therefore

$$\left(1 - \frac{b}{x}\right)^4 = 1 - \frac{4b}{RA} \qquad (79)$$

This equation has four solutions. Two are pure imaginary and can be discarded. From the two real solutions the one which is physically relevant is

$$x = \frac{b}{1 - \left(1 - \frac{4b}{RA}\right)^{\frac{1}{4}}} \tag{80}$$

This is the exact position of point S.

It is important to note that Diderot does not use his previous results (Propositions I and II) in order to obtain the position of point S. The fact is that he does not need to: Newton's result follows from the simple assumption that retardation is proportional to the arc. But since Diderot wrote also his article arguing for a retardation proportional to the square of the arc, that is

$$Rr = \alpha(RA)^2, \tag{81}$$

why didn't he bother to write down the equation that would replace (71) and solved it? The new equation can be written in a straightforward manner, albeit after a very long algebraic manipulation. One obtains

$$4bx^{30} - 4a^2bx^{28} + 12\,a^3b^2x^{26} - 30\,a^4b^3x^{24} + 64\,a^5b^4x^{22} - 118\,a^6b^5x^{20} +$$
$$188\,a^7b^6x^{18} - 258\,a^8b^7x^{16} + 302\,a^9b^8x^{14} - 298\,a^{10}b^9x^{12} + 244\,a^{11}b^{10}x^{10} -$$
$$162\,a^{12}b^{11}x^8 + 84\,a^{13}b^{12}x^6 - 32\,a^{14}b^{13}x^4 + 8\,a^{15}b^{15}x^2 - a^{16}b^{15} = 0 \tag{82}$$

where, for the sake of clarity, we replaced RA by the letter a. We don't know if Diderot ever wrote this equation but in any case it does not appear in the memoir. This is not surprising and it is quite pointless trying to find the roots to this equation. We can however try to find an approximate solution of the reduced polynomial: if we, with Diderot, consider b to be small, we can discard the higher powers of b and keep only the terms to lowest order, that is

$$4b\,x^{30} - 4a^2b\,x^{28} + 12\,a^3b^2\,x^{26} = 0 \longrightarrow x^4 - a^2x^2 + 3\,a^3b = 0 \tag{83}$$

This equation can be trivially solved to give the roots

$$x = \pm\frac{a}{\sqrt{2}}\sqrt{1 \pm \sqrt{1 - \frac{12b}{a}}} \tag{84}$$

Again, approximating the square root as in Eq. (47) one gets a physically relevant solution in the form

$$x = a - \frac{3}{2}b \tag{85}$$

Remembering that a is our short notation for AR this solution can be written as

$$x = RA - \left(b + \frac{1}{2}b\right) \tag{86}$$

which is the same approximate solution Eq. (75] that Newton got in the case of a retardation proportional to the arc. To conclude, in the limit of small amplitude oscillations, where velocities are small, replacing a linear by a quadratic drag makes no significant difference. This is what one observes in the experiments discussed in the last section of this article: the changes in period due to linear and quadratic drag

are of the same order of magnitude and it would have been impossible for Diderot or Newton to detect those.

Another interesting point worth noticing is the fact that Diderot did not apply the method he develops for the case of a linear drag. He must have been aware that this would imply replacing the integral in Eq. (22) by

$$\int_b^x \frac{\sqrt{2pb - 2px'}\, a\, dx'}{\sqrt{2ax' - x'^2}},\qquad(87)$$

which can be solved only numerically [14].

5 The Mathematical Pendulum from a Modern Perspective

5.1 The Problem and the Solution

The mathematical (ideal) pendulum is one of the most paradigmatic models of classical mechanics. It consists of a point like mass m attached to a frictionless point O through an ideal (massless and inextensible) string of length l. As it swings, the position of the bob can be described, for any given instant t, by the angle $\theta\,(t)$ measured relative to its rest position A (see Fig. 5.1). Even though one can write Newton's
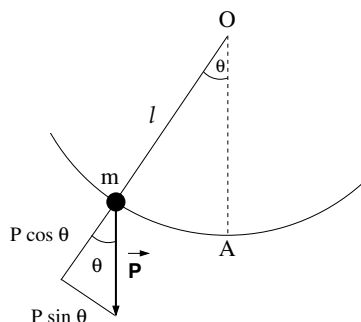
Fig. 8. The ideal or mathematical pendulum. A point like mass m attached to a point O through an ideal string of length l. θ is the angle the bobs makes with respect to the vertical OA. The angle θ is positive if to the right of the vertical OA and negative to the left.

equation of motion for the displacement $d\mathbf{s}$ along the arc, it is more convenient to write the same equation in terms of the angle θ $(ds = l\, d\theta)$

$$m\, l\, \frac{d^2\theta}{dt^2} + m\, g\, \sin\theta = 0\qquad(88)$$

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[14] It can be recast in terms of a very complicated expression involving an elliptic integral of the second kind, whose values can then be looked up in a table or solved numerically. Elliptic integrals go back to A.-M. Legendre's (1853 − 1833) and N.H. Abel's (1802 − 1829) works of 1825 and 1823, respectively (Good 2001).

This nonlinear differential equation has an implicit solution for θ as a function of time t in terms of Legendre's elliptic integral of the first kind $F(k, \psi)$ (Függe 1967):

$$\sqrt{\frac{g}{l}}\, t = \int_0^\psi \frac{d\psi\,'}{\sqrt{1 - k^2 \sin^2 \psi\,'^{\,2}}} = F(k, \psi) \tag{89}$$

The angular displacement θ is related to ψ through

$$\sin\frac{\theta}{2} = k \sin \psi, \tag{90}$$

where k is a quantity related to the amplitude θ_0 through

$$k = \sin\frac{\theta_0}{2} \tag{91}$$

As the bob is let loose from θ_0, it will swing and by solving the above integral numerically, one can determine the value of $\theta(t)$ at any given time t. One is normally interested in the period T of one complete oscillation. From the result above one may easily obtain

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\psi\,'}{\sqrt{1 - k^2 \sin^2 \psi\,'^2}} = 4\sqrt{\frac{l}{g}}\, K(k) \tag{92}$$

where $K(k) = F(k, \pi/2)$ is the complete elliptic integral of the first kind. This result follows by remembering that a period corresponds to the time it takes the bob to return to θ_0 after its release. In this case $\theta(T) = \theta_0$ implies $\psi = \pi/2$ in Eq. (90) and hence $F(k, \psi) \to F(k, \pi/2) = K(k)$.

This solution is rather involved and what one will usually find in physics textbooks is the small amplitude approximation: in the case of small θ, one may replace $\sin \theta \approx \theta$ in Eq. (88) and obtain a *linear* differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \tag{93}$$

which can be easily solved to give (for the initial condition $\theta(t = 0) = \theta_0$)

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}\, t\right) \tag{94}$$

The period T is then given by

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{95}$$

The relevance of this exact solution in the small-θ limit lies not only in its use as a didactic tool in the study of differential equations. It shows that for small angles, the period of the pendulum Eq. (95) depends only on the length l of the string and the acceleration of gravity g and not on the amplitude of the swing. This makes the small-amplitude pendulum an ideal time-keeping device.

By measuring the period and the length of the pendulum, one may also use Eq. (95) to find the acceleration of gravity g with quite good precision, and this method was the preferred one before being substituted by direct measurements on free-falling bodies (Nelson and Olsson 1981).

The small-amplitude approximation can be obtained from the general solution Eq. (92) by rewriting it as a power series in $k = \sin(\theta/2)$

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{4}\,k^2 + \frac{9}{64}k^4 + \ldots\right) \qquad (96)$$

For an amplitude of $\theta_0 = 20\,°$ ($k = 0.1736$) the correction that Eq. (96) introduces amounts to about 0.7% as compared to Eq. (95) while for $\theta_0 = 40\,°$ ($k = 0.3420$) it amounts to about 3%. At the time Diderot wrote his memoirs the solution of the pendulum equation for arbitrary angles was not known.

5.2 The Effect of Air Resistance

The simplicity of the approximate solution is quite deceptive not because the small angle approximation is unphysical – for this one may always go back to the entire solution - but because in real applications the ideal conditions assumed from the onset are not valid: bobs are not point like masses, strings are not massless and inextensible and damping by air and friction at the pivoting point do play a role. The bob will eventually stop swinging if there is no external force to keep it moving. For the present work, the most relevant source of damping is the resistance caused by the surrounding air. Diderot, as did Newton before him, considered the effect of air resistance on a spherical bob, in spite of the fact that it also acts on the wire from which the bob hangs (see discussion below). The main question that puzzled physicists for a long time and can be only effectively dealt in a phenomenological manner is how the drag (as the force due to air resistance is usually called in technical parlance) depends on the relative speed between the moving body and the surrounding air. This is the main point of divergence between Diderot and Newton and the correct answer to this question is not or mere academic or historical interest: it has consequences that go beyond the problem discussed here, as for instance in the design of aircraft wings or of any object that moves through air.

We know today a lot more about the effects of drag than Diderot (or for that matter Newton) knew at the time he wrote his memoir. Hydrodynamics was still on the making and ideas and techniques which allowed one to handle the effect of air resistance was developed mostly during the 18th and 19th centuries by people like G. G. Stokes (1819 – 1903), J. W. Struth (Lord Rayleigh, 1842 – 1919), O. Reynolds (1842 – 1912) and L. Prandtl (1875 – 1953). What determines the type of drag acting on any part of a system is determined by the Reynolds number characteristic of that part (for the pendulum, these would be the bob and the string). This dimensionless quantity was first introduced by Stokes in 1858 (Stokes 1858) to predict flow patterns in fluids but was named after Reynolds, who popularized its use in 1883 (Reynolds 1883). The Reynolds number is defined as

$$Re := \frac{\rho v L}{\eta} \qquad (97)$$

where ρ and η represent the density and dynamic viscosity of the surrounding medium respectively and v is a characteristic velocity, usually the mean relative velocity between fluid and body. L is a characteristic length of the body, which in the case of the bob would be its diameter.

The Reynolds number is the key as to whether the drag be Stokes-like or Newton-like. Low Reynolds numbers, up to $Re \sim 10$, imply that the flow past the body will be laminar (not turbulent) and will cause a drag of the Stokes type. This is the case

for instance of a ball falling through honey (low velocity, high viscosity), where it quickly reaches a terminal velocity and then falls at constant speed. High values of Re, of order $\sim 10^3$ to 10^5 (high velocity, low viscosity) corresponds to a drag force which follows Newton's v^2–law. As the pendulum consists of two parts, a string and a bob, in a real experimental setup one has to treat each component according to its own Reynolds number. As an example, in the experiments conducted by Nelson and Olsson, the string had a Reynolds numbers of 6 while the bob had an Re of 1100. This implies that the best fit to the problem of a swinging bob would be (Nelson and Olsson 1981):

$$F(v) = a\,|v| + b\,v^2 \tag{98}$$

where a, b are adjustable parameters and the first term on the right-hand side accounts for the drag on the string while the second for the drag on the bob. For the experiment just mentioned, a is usually one order of magnitude smaller than b, so by choosing a string thin enough, one would not be too far off the mark if one just considered $a = 0$ and took $F(v) = b\,v^2$, as Diderot did.

Another point that makes matters significantly more difficult as was not explicitly discussed in the previous literature on Diderot is the fact that, as the bob swings, its velocity changes and so does its Reynolds number. The usual heuristic approach to deal with this problem is to consider a generalization of Eq. (98) in the form

$$F = \frac{1}{2}\,C_D\,A\rho\,v^2 \tag{99}$$

where the dimensionless number C_D, known as the drag coefficient, incorporates the effect of a changing Reynolds number. In the expression above A is an area associated with the moving body. C_D is a function of the Reynolds number and is determined by adjusting experimental data for F as a function of v. For values of Re of the order of 1 or smaller, C_D is inversely proportional to Re, that is $C_D \sim Re^{-1}$ while for high values of Re, C_D is a constant. This way one tries to capture the whole range of regimes under one single equation. For the case of a spherical bob, an expression for C_D accurate to within 10% for values or Re over the range $0 \le Re \le 2 \times 10^5$ can be found in White's book (White 1974) and is given by

$$C_D \simeq \frac{24}{Re} + \frac{6}{1 + Re^{1/2}} + 0.4 \tag{100}$$

where the first term on the right-hand side accounts for Stokes's law, the last for Newton's v^2 law and the middle term for the transition between both regimes.

So the question about a v or v^2–dependence is not straightforward, as already pointed out by previous authors (Krakeur and Krueger 1941; Coolidge 1949). However, by assuming an drag of the type given by Eq. (99), Diderot sounds surprisingly modern. What about Newton? Even though Newton he does not refer to the size of the bob he used in trying to prove the Third Law, in Book II he is explicit about the size of pendulums he experimented with: a wooden bob of approximate diameter of $d = 17.46\ cm$ and mass $m = 1625\ g$ and a leaden bob of $d = 5.08\ cm$ and $m = 744.2\ g$. This would imply that Newton's bobs had a range of Re from approximately 1100 to 3700, which calls for a v^2–law, assuming that he used the same bob sizes in his collision experiments. Coincidentally his second bob is about the same size and mass as the one used by Nelson and Olsson in their experiments, so we can use their results to see how far off Diderot or Newton might have been (Nelson and Olsson 1981).

What the experimental results show is that the corrections are of the same order of magnitude, irrespective of whether one considers the first type of force (Newton) or the second (Diderot). In their experiments, Nelson and Olsson took an initial amplitude of $3°\pm0.3°$ which introduces a finite-amplitude correction of 596 μs when

compared to the ideal period. In the case of linear damping the correction to the period, discounting the finite-amplitude correction, was of the order of 0.033 μs. For the quadratic and one it amounted to 0.027 μs. The difference is negligible and it would have been impossible for Diderot (or Newton) to detect those [15]. This is reflected in the equality of Diderot's solution for Newton's S, Eq. (75) in the linear case and our solution Eq. (86) in the quadratic case.

5.3 The Small-Angle Approximation and Quadratic Damping: the Method of Lindstedt-Poincaré

Unbeknown to Diderot, he was trying his hand at a problem whose exact solution still eludes us. If one looks up any textbook on the effect of air resistance on a pendulum – and in most textbooks the pendulum equation means the linearized version (93) and not Diderot's nonlinear Eq. (88) – one will always find a *linear* drag force $F_R(v)$ and not a quadratic one. To the author's knowledge, none of the texts consulted give any justification, experimental or otherwise, as to why this should be so. The reason might be purely didactic: if one considers a drag force $F_R(v) = -\gamma v$, Eq. (93) becomes

$$\frac{d^2\theta}{dt^2} + \frac{\gamma}{m}\frac{d\theta}{dt} + \frac{g}{l}\,\theta = 0 \tag{101}$$

for which one may easily find an exact analytical solution with an exponentially damped amplitude (Flügge 1967). This is not the case of quadratic damping

$$\frac{d^2\theta}{dt^2} - \frac{\gamma\,l}{m}\left(\frac{d\theta}{dt}\right)^2 + \frac{g}{l}\,\theta = 0 \;, \tag{102}$$

in which case there is no exact solution anymore. What is worse, the equation is not even analytic because the sign of the force (and hence the equation) must be adjusted each half-period to guarantee that the damping force always acts as to retard the pendulum's movement. It took a century after Diderot's death for A. Lindstedt (1854 – 1939) and H. Poincaré (1854 – 1912)to independently develop a perturbative approach. We present this approach next (Nelson and Olsson 1981; Linz 1995; Cvetićanin 2009).

The difficulty with Eq. (102) is that standard perturbation methods will not work. This is because there are two time scales involved, the one associated with the period of the pendulum and the other with dissipation. A standard perturbation method leads to the appearance of so-called secular terms, which are terms which grow with time, whereas one knows that the solution has to be periodic. The Lindstedt-Poincaré method is a way of removing these secular terms when dealing with weakly nonlinear problems with periodic solutions.

We consider Eq. (102) for a half-period of oscillation, since the solution obtained can be reapplied to other half-periods. We rewrite this equation as

$$\ddot{\theta} - \epsilon\,\dot{\theta}^2 + \omega_0^2\,\theta = 0 \tag{103}$$

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[15] In the experiment one does not measure the effect directly. One considers a drag of the type $F(v) = a\,|v| + b\,v^2$ and finds the best values of a and b that fit the data. From that one can infer the retardation effect using the approximate solutions with the fitted values. There is an added complication, since as the pendulum swings and is damped, the amplitude changes and consequently the period. To measure the effect of air damping one has to average over many oscillations and discount the finite amplitude correction accumulated during swings.

where $\dot{\theta} = d\theta/dt$, $\epsilon = (\gamma\, l/m)$ and $\omega_0^2 = g/l$. We want to find a solution with a period $T = 2\pi/\omega$. One introduces a new variable

$$\phi = \omega t \tag{104}$$

in terms of which Eq. (103) can be written as

$$\omega^2 \theta'' - \epsilon\omega^2 {\theta'}^2 + \omega_0^2\, \theta = 0 \tag{105}$$

where now θ' stands for $d\theta/d\phi$. The following step is to write θ and ω in terms of a series expansion in the small parameter ϵ

$$\theta = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \cdots$$
$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots \tag{106}$$

and then substitute (106) into (105). Setting the factors of each power of ϵ equal to zero, we obtain, through order ϵ^2, the following set of equations

$$\psi_0'' + \psi_0 = 0$$
$$2\left(\frac{\omega_1}{\omega_0}\right)\psi_0'' + \psi_1'' + \psi_1 - {\psi_0'}^{\,2} = 0$$
$$\left[2\left(\frac{\omega_2}{\omega_0}\right) + \left(\frac{\omega_1}{\omega_0}\right)^2\right]\psi_0'' + 2\left(\frac{\omega_1}{\omega_0}\right)\psi_1'' +$$
$$\psi_2'' + \psi_2 - 2\left(\frac{\omega_1}{\omega_0}\right){\psi_0'}^{\,2} - 2\psi_0'\,\psi_1' = 0 \tag{107}$$

These can be solved recursively. The solution of the first equation with $\psi_0 = \theta_0$ and $\psi_0' = 0$ at $\phi = \omega t = 0$ is

$$\psi_0 = \theta_0 \cos\phi \tag{108}$$

Substituting this into the second equation in (107) leads to

$$\psi_1'' + \psi_1 = 2\left(\frac{\omega_1}{\omega_0}\right)\theta_0 \cos\phi + \frac{1}{2}\theta_0^2 \sin^2\phi \tag{109}$$

The first term on the right hand side contributes to a term of the form

$$\left(\frac{\omega_1}{\omega_0}\right)\theta_0\phi \sin\phi \sim t \sin t \tag{110}$$

which is secular, i.e. increases without bound. As we are looking for periodic solutions we must then have

$$\omega_1 = 0 \tag{111}$$

So, the solution to (109) that satisfies the initial conditions $\psi_1 = 0$ and $\psi_0' = 0$ at $\phi = \omega t = 0$ is

$$\psi_1 = \frac{1}{6}\theta_0^2(3 - 4\cos\phi + \cos 2\phi) \tag{112}$$

where we have used the identity $\sin^2\phi = (1/2)(1 - \cos 2\phi)$. From this result it follows that at the end of the first half-cycle ($\phi = \pi$) the amplitude will be

$$\theta_1 = -\theta_0\left(1 - \frac{4}{3}\epsilon\theta_0\right) \tag{113}$$

This method can be applied successively to find the amplitudes of the next half-cycles. From this result it follows that the difference between the first two successive amplitudes will be

$$\theta_0 - |\theta_1| = \frac{4}{3}\epsilon(\theta_0)^2 \tag{114}$$

If we translate Eq. (114) into the language of arcs traversed by the bob, it is telling us that the difference in arc is proportional to the square of the arc traversed by the bob during its descent. Apart from the prefactor of 4/3, this is the conclusion Diderot arrived at in his memoir.

6 Conclusions

In 1748 Diderot published a series of memoirs on different subjects of Mathematics. In the Fifth Memoir, he studied the effect of air resistance on the movement of the pendulum when this drag is proportional to the square of the bob's velocity. Since Diderot quotes a passage of Newton's *Principia* where this problem is discussed considering a resistance linear in the velocity, it has been argued in the past that the sole purpose of Diderot was to correct an assumption that Newton made and Diderot thought incorrect. In the present article it has been argued that Diderot's memoir may have served a different purpose: a careful analysis of his methods shows that Diderot wrote his memoir as a detailed guide of how to obtain the results Newton presented without further justification and may have been part of a larger plan of publishing his own commented edition of the *Principia*. In this he was thwarted by Le Seur and Jacquier, who published their 3-volume annotated edition between 1739 and 1742.

By assuming the drag to be quadratic in the bob's velocity, he obtained a difference in amplitude (retardation) between swings quadratic in the displacement, a result which is confirmed by a Lindstedt-Poincaré analysis of the same problem. By considering a drag linear in the velocity, he shows that Newton's results can be obtained from a more general solution in the limit of weak drag.

The question of whether the drag should be linear or quadratic has been discussed in detail. If one considers the problem from a modern perspective, the Reynolds number associated with a spherical bob of the size Newton used imply that drag should be quadratic, thus confirming Diderot's assumption. However, from a practical point of view, since the velocity varies during swings and one is usually interested in small amplitude oscillations, the difference in results obtained in either case is beyond the precision that Newton had at his disposal and would not have been detected by Diderot in case he had conducted himself the experiments.

Diderot handled a full problem for which even the simplified version (small-amplitude approximation with quadratic drag) had to wait 100 years to be dealt with perturbative methods. We may agree with Coolidge when he says that (Coolidge 1949)

> '... Diderot had hold of a problem that was too much for him.'

However, as Eq. (114) shows, this should not diminish his merit: he obtained the same functional dependence on the retardation as one would get using the modern perturbative approach by means of a first-order approximation to solve an integral equation. He was no professional mathematician, as his fellow *philosophe* d'Alembert, but his involvement went beyond that of a simple amateur. His delving into the fields of mathematics and physics speak for themselves.

Appendix: Original Quotations

Diderot 1774, p. 580: *'On lui met entre les mains des cahiers d'arithmétique, d'algèbre et de géométrie qu'il dévora. Entraîné par la suite à des études plus agréables, il se plut à la lecture d'Homère, de Virgile, du Tasse e de Milton, mais revenant toujours aux mathématiques, comme un époux infidèle, las de sa maîtresse, revient de temps en temps à sa femme.*

Diderot 1748, p. 320 : *'Si l'endroit òu Newton calcule la résistance que l'air fait au mouvement d'un pendule vous embarasse, que votre amour-prope n'en soit point affligé. Il y a, vous diront les plus grands géométres, dan s la profondeur et lat laconicité des Principes mathématiques, de quoi consoler partout un homme pénétrant qui aurait quelque peine à entendre; et vous verrez bientôt que vous avez ici pour vous une autre raison que me paraît encore meilleure; d'est que l'hypothèse d'où cet auteur est parti n'est peut-être pas exacte.'*

Diderot 1748, p. 332: *'J'ai pour Newton toute la déférence qu'on doit aux hommes unique dans leur genre'.*

Diderot 1748, pp. 320-321: *'Mais une chose me surprend; c'est que vous vous soyez avisé de vous adresser à moi, pour vous tirer d'embarass. Is est vrai que j'ai étudié Newton, dans le bassein de l'éclaircir; je vous avouerai même que ce travail avait été poussé, sinon avec beaucoup de succès, du moins avec assez de vivacité; mais je n'y pensais plus dès le temps que les RR Pères Le Sueur et Jacquier donnèrent leur Commentaire; et je n'ai point été tenté de le rependre. Il y aurait eu, dans mon ouvrage, fort peu des choses qui ne soient dans celui des savants géomètres; et il y en a tant dans le leur, qu'assurément on n'eût pas rencontrées dans le mien! Qu'exigez-vous de moi? Quand les sujets mathématiques m'auraient été jadis très-familiers, m'interroger aujourd'hui sur Newton, c'est me parler d'un rêve de l'an passé. Cependant, pour persévérer dans l'habitude des vous satisfaire, je vais, à tout hazard, feuilleter mes paperasses abandonnées, consultes les lumières de mes amis, vous communiquer ce que j'en pourrai tirer, et vous dire, avec Horace: Si quid novisti rectius istis, candidus imperti. Si non, his utere mecum'.*

Diderot 1748, p. 321: *'Soit un pendule M qui décrit dans l'air l'arc BA, étant attaché à la verge GM fixe en G. On demande la vitesse de ce pendule en un point quelconque M, en supposant qu'il commence à tomber du point B.'*

Diderot 1748, p. 323: *'Dans cette équation, je mets, au lieu du petit arc Mm sa valeur $-\frac{a}{\sqrt{a^2-b^2}}\,dx$, avec le signe −, parce que v croissant à mesure que le pendule descend, x diminue au contraire.'*

Diderot 1748, p. 324: *'Supposons qu'un pendule, placé dans la situation vertical GA, reçoive une impulsion ou vitesse h suivant l'horizontale AR. On demande sa vitesse en un point quelconque M'.*

Diderot 1748, p. 326: *'Si le pendule A est un petit globe, la résistance f, toutes choses d'ailleurs égales, es en raison inverse du diamètre de ce globe et de sa densité; car la résistance de l'air à deux globe de différents diamètres est comme le surface ou le carré des diamètres; et cette résistance doit être divisée par la masse, laquelle est comme la densité multipliée par le cube du diamètre. Donc l'arc Ck, toutes choses d'ailleurs égales, est come AB^2 divisé par le produit du dimatètre du globe et de sa densité.'*

Le Seur and Jacquier 1739, p. 37: *'Trahatur corpus A, ad arcus EAF, punctum quodvis R, et demittatur inde, sublata medii resistentia ad eandem altitudinem M, ascendere et rursus ad punctum R, redire debet. Cum autem post unam oscillationem exitu et reditu compositam perveniat (ex hyp.) ad punctum V arcus RV exponet medii retardationem in duplici ascensu et descensu; quare ut habeatur medii retardatio in uno tantum descensu, sumenda est quarta pars totius retardationis, id est quarta pars arcus RV, dummodo ille descensus neque ex puncto supremo R, neque ex infimo V ordiatur: nam cum major sit medii retardatio in arcu majori quam in minori semperque fiant minores arcus a pendulo oscillante descripti, inaequales quoque erunt retardationes in singulis arcubus, et retardatio descensus per RA, major erit quarta parte totius retardationis RV ut retardatio ultimi ascensus AV, minor erit quarta parte totius retardationis RV. Hoc autem aut simili calculo determinavit Newtonus punctum S tale ut retardatio in descensu per SA sit quarta pars totius retardationis RV. Dicatur arcus RA, 1, arcus RV, 4b, arcus quaesitus SA, x; sintque retardationes arcubusdescriptis proportionales, erit arcus SA (x) ad arcum RA (1) ut retardatio arcus SA quae statuitur esse b, seu quarta pars totius RV, ad retardationem primi arcus RA quae erit b : X. Quaerantur successive retardationes secundi, quartive arcus eadem ratione ; arcus autem secundus est equalis primo RA, dempta ejus retardationes b : X. Tertius arcus aequalis secundo dempta ejus retardatione, et sic deinceps, omnes vero illae retardationes simul sumptae aequabuntur toti retardationi RV seu 4b ; unde fit aequatio ex qua valor arcus SA, seu x, obtinebitur, per approximationem autem invenietur aequalis 1 3/2b sumatur itaque RS aequalis quartae parti cum ejus semisse totius retardationis RV, retardatio per arcum SA erit aequalis ST quartae parti totius RV, ideoque cadat corpus ex puncto S, ejus celeritas in A eadem est sine errore sensibili, ac si in vacuo decidisset ex T'.*

References

d'Alembert, J. 1747. *Recherches sur la courbe que forme une corde tendue mise en vibration*, Mem. Acad. Sci. Berlin **3**, pp. 214 – 229.

d'Alembert, J. 1780. *On the Calculus of Probabilities*, Opuscules Mathématiques, Vol. II, 52nd Memoir, pp. 39 - 60. Editions CNRS, Paris, 2011.

d'Alembert, J. 1761. *Reflections on the Calculus of Probabilities*, Opuscules Mathématiques, Vol. II, 10th Memoir, pp. 1 - 25. Editions CNRS, Paris, 2011.

d'Alembert, J. 1768. *Extracts of Letters on the Calculus of Probabilities and on the Calculus Related to Inoculation*, Opuscules Mathématiques, Vol. V, 27th Memoir, pp. 283 - 310. Editions CNRS, Paris, 2011.

d'Alembert, J. 1768. *On the Analysis of Games*, Opuscules Mathématiques, Vol. IV, 23rd Memoir, pp. 79 - 92. Editions CNRS, Paris, 2011.

d'Alembert, J. 1784/. *Croix ou Pile*. In *Encyclopédie, ou Dictionaire Raisonné des Sciences, des Arts et des Metiers*, Vol. 4, pp. 512 – 513. Paris.

Ballstadt, K. 2008. *Diderot: Natural Philosopher*, SVEC 2008:09, Oxford, Voltaire Foundation.

Bessel, F.W. 1828. *Untersuchungen über die Länge des einfachen Secundenpendels*, Druckerei der Königlichen Akademie der Wissenschaften, Berlin (reprint Verlag von Wilhelm Engelmann, Leipzig, 1889).

Cannon, J.T. and Dostrovsky S., 1981 *The Evolution of Dynamics: Vibration Theory from 1687 to 1742*, Springer Verlag, New York.

Coolidge, J.L. 1949. *The Mathematics of Great Amateurs*, 2nd edition, Clarendon Press, Oxford, 1990, pp. 178 – 185.

Cvetićanin, L. 2009. *Oscillator with strong quadratic damping force*, Publications de l'Institute Mathématique, Nouvelle série, tome 85 (99), 119 – 130.

Dahmen, S.R. 2014. *The Mathematics and Physics of Diderot. On Involutes*, in preparation.

Darrigol, O. 2005 *Worlds of Flow: A History of Hydrodynamics from the Bernoullis to Prandtl*, Oxford University Press, chapter 7.

Daston, L.J. 1979. *D'Alembert's Critique of Probability Theory*, Historia Mathematica **6**, pp. 259 - 279.

Darrigol, O. and Frisch, U. 2009. *From Newton's Mechanics to Euler's Equations*, Physica D: Nonlinear Phenomena **237** (14-17), pp. 1855 - 1869 and references therein.

Diderot, D. 1748. *Mémoires sur différens sujets de Mathématiques*, Durant et Pissot, Paris. Reprinted in *Oeuvres complètes : édition critique et annotée. V. 2: Philosophie et mathématique: Idées I.* R. Niklaus *et. al.* (eds), Herrmann, Paris, 1975, pp. 231 - 338.

Diderot, D. 1761. *Noveaux Mémoires sur Différents Sujets de Mathématiques* in *Oeuvres complètes : édition critique et annotée. V. 2: Philosophie et mathématique: Idées I.* R. Niklaus *et. al.* (eds), Herrmann, Paris, 1975, pp. 339 - 361.

Diderot, D. 1774. *Refutation suivie de l'ouvrage d'Helvetius intitule l'homme*, Éditions Gallimard, Paris, 2010.

Diderot, D. 1975. *Prèmieres notions sur les mathématiques a l'usage des enfants ou premier livre classique du 1er cours d'études* in *Oeuvres complètes : édition critique et annotée. V. 2: Philosophie et mathématique: Idées I.* R. Niklaus *et. al.* (eds), Herrmann, Paris, 1975, p. 367.

Euler, L. 1739. *Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae.* Petropoli (St. Petersburg), ex Typographia Academiae Scientiarum. Reprinted in Smith, C.S. 1960. *Leonhard Eulers Tentamen novae theoriae musicae. A Translation and Commentary.* Indiana University Press, Bloomington.

Flügge, S. 1967. *Lehrbuch der Theoretischen Physik Band II*, Springer Verlag, Heidelberg, pp. 49 – 53.

Fourier, J. 1807. *Mémoire sur la propagation de la chaleur dans les corps solides*, Nouveau Bulletin des sciences par la Société Philomatique de Paris I (6). Paris: Bernard. March 1808, pp. 112116; Reprinted in Oeuvres complètes, tome 2., Paris, 1890, pp. 215221.

Gauld, C.F. 2009. *Newtons Use of the Pendulum to Investigate Fluid Resistance: A Case Study and some Implications for Teaching About the Nature of Science*, Science and Education **18**, pp. 383 – 400.

Gauld, C.F. 2010. *Newtons Investigation of the Resistance to Moving Bodies in Continuous Fluids and the Nature of 'Frontier Science'*, Science and Education **19**, pp. 939 – 961.

Gay, P. 1966. *The Enlightenment: An Interpretation. The Rise of Modern Paganism*, Vintage Books, New York, p. 14.

Good, R.H. 2001. *Elliptic Integrals, the forgotten functions*, Eur. J. Phys **22**, pp. 119 – 126.

Huygens, C. 1673. *Horologium Oscillatorium: sive de motu pendulorum ad horologia aptato demonstrationes geometricae.* Reprinted in Culture et Civilisation, Bruxelles, 1966.

Kleiner, I. 1989 *Evolution of the Function Concept: a Brief Survey*, The College Mathematics Journal **20/4**, pp. 282 – 300.

Krakeur, L.G. and Krueger, R.L. 1941. *The Mathematical Writings of Diderot*, Isis **33/2**, pp. 219 – 232.

Lindstedt, A. 1882. *Beitrag zur Integration der Differentialgleichungen der Störungstheorie*, Abh. K. Akad. Wiss. St. Petersburg 4, p. 31.

Linz, S.J. 1995. *A simple nonlinear system: the pendulum with quadratic friction*, Eur. J. Phys. **16**, pp. 67 – 72.

Loria, G. 1930. *Curve piane speciali*, vol. II, p. 125. Milano.

Matthews, M.R. Gauld, C.F. and Stinner, A. 2005 *The Pendulum: Scientific, Historical, Philosophical and Educational Perspectives*, Springer, pp. 139 – 150.

Mayer, J. 1975. *Oeuvres Completes de Diderot: Oeuvres Mathématiques*, vol. 2, Hermann, Paris.

Nelson, R.A. and Olsson, M.G. 1986. *The pendulum – rich physics from a simple system*, Am. J. Phys. **54/2**, pp. 112 – 121.

Newton, I. 1687 *Principia Mathematica Philosophiae Naturalis*. Translated by A. Motte as *Mathematical Principles of Natural Philosophy*, University of California Press, Los Angeles.

Plana, J. 1835. *Mémoire sur le Mouvement d'un Pendule dans un Millieu Résistant*, Turin, L'Imprimérie Royale. Credited as Jean Plana.

Poincare, H. 1893. *Les Méthodes Nouvelles de la Mécanique Céleste II*, Gauthier-Villars Imprimieus Librarie, Paris, & 123 - 128.

Rameau, J.-P. 1722. *Traité de l'harmonie réduite à ses principes naturels*, Ballard, Paris; reprint ed. in Jean-Philippe Rameau [1683-1764] Complete Theoretical Writings, Miscellanea, vol. 1 American Institute of Musicology, 1967, Münster.

Ravetz, J.R. 1961 *Vibrating Strings and Arbitrary Functions*, in *The Logical of Personal Knowledge: Essays presented to M. Polanyi on his Seventieth Birthday*, The Free Press, 1961, pp. 71 – 88.

Reynolds, O. 1883. *An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels*, Philosophical Transactions of the Royal Society 174, pp. 935-982.

Samuelson, P.A. 1977. *St. Petersburg Paradoxes: Defanged, Dissected, and Historically Described*, J. of Economic Literature, **15** (1), pp. 24 - 55.

Schulze. M. 1782. *Sur les horloges a pendule. Premier Mémoire*, Noveaux Mémoire de l'Académie Royale des Sciences e Belles Lettres (Königliche Akademie der Wissenschaften Berlin), Decker, pp. 349 – 350; *Sur les horloges a pendule. Seconde Mémoire*, ibid, pp. 359 – 375.

Smith, G.E. 1998. *Newton's study of fluid dynamics*, International Journal of Engineering Science **36**, pp. 1377 – 1390 and references therein.

Swijtink, Z.G. 1986. *D'Alembert and the Maturity of Chances*, Stud. Hist. Phil. Sci. **17** (3), pp. 327 – 349.

Le Seur, T. and F Jacquier, F. 1822 *Isaaci Newtoni Philosophiae Naturalis Principia Mathematica, Perpetuis Commentariis Illustrata*, Vol. I, Glasgow, 1822, pp. 37-39.

Stokes, G.G. 1851. *On the Effect of the Internal Friction of Fluids on the Motion of Pendulums*, Transactions of the Cambridge Philosophical Society 9, pp. 8–106.

Taylor, B., 1713 *De motu nervi tensi*, Phil. Trans. London 28, p. 26 – 32.

Truesdell C., 1960. *The Rational Mechanics of Elastic or Flexible Bodies, 1638 – 1788. Introduction to vols. X and XI of Leonhardi Euleri Opera Omnia, Seriei Secundae*, Zürich, pp. 129 – 281.

Virtanen, H. 2006. *Studies of Earth Dynamics with the Superconducting Gravimeter*, PhD Thesis. Publication 133 of the Finnish Geodetic Institute, Helsinki

Wagner, E. 1867. *Mémoire sur le Pendule et le Balancier considéré comme Régulateurs des Instruments a Reguler le Temps*, Typographie de Ch. Maréchal, Paris.

White, F.M. 1974. *Viscuous Fluid Flow*, McGraw-Hill, New York, p. 209.

Wilson, A.M. 1972. *Diderot*, Oxford University Press, Oxford.

Wolf, C. 1889. *Mémoires Sur Le Pendule, Précédés D'une Bibliographie*, Gauthier-Villars, Paris.